

BOOTSTRAP-BASED K -SAMPLE TESTING FOR FUNCTIONAL DATA

Efstathios PAPARODITIS* and Theofanis SAPATINAS†
 Department of Mathematics and Statistics, University of Cyprus,
 P.O. Box 20537, CY 1678 Nicosia, CYPRUS.

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Abstract

We investigate properties of a bootstrap-based methodology for testing hypotheses about equality of certain characteristics of the distributions between different populations in the context of functional data. The suggested testing methodology is simple and easy to implement. It bootstraps the original functional dataset in such a way that the null hypothesis of interest is satisfied and it can be potentially applied to a wide range of testing problems and test statistics of interest. Furthermore, it can be utilized to the case where more than two populations of functional data are considered. We illustrate the bootstrap procedure by considering the important problems of testing the equality of mean functions or the equality of covariance functions (resp. covariance operators) between two populations. Theoretical results that justify the validity of the suggested bootstrap-based procedure are established. Furthermore, simulation results demonstrate very good size and power performances in finite sample situations, including the case of testing problems and/or sample sizes where asymptotic considerations do not lead to satisfactory approximations. A real-life dataset analyzed in the literature is also examined.

Some key words: BOOTSTRAP; COVARIANCE FUNCTION, FUNCTIONAL DATA; FUNCTIONAL PRINCIPAL COMPONENTS; KARHUNEN-LØVE EXPANSION; MEAN FUNCTION; K -SAMPLE PROBLEM.

1 INTRODUCTION

Functional data are routinely collected in many fields of research; see, e.g., Bosq (2000), Ramsay & Silverman (2002, 2005), Ferraty & Vieu (2006), Ramsay *et al.* (2009) and Horváth & Kokoszka (2012). They are usually recorded at the same, often equally spaced, time points, and with the same high sampling rate per subject of interest, a common feature of modern recording equipments. The estimation of individual curves (functions) from noisy data and the characterization of homogeneity

*Email: stathisp@ucy.ac.cy

†Email: fanis@ucy.ac.cy

and of patterns of variability among curves are main concerns of functional data analysis; see, e.g., Rice (2004). When working with more than one population (group), the equality of certain characteristics of the distributions between the populations, like their mean functions or their covariance functions (resp. covariance operators) is an interesting and widely discussed problem in the literature; see, e.g., Benko *et al.* (2009), Panaretos *et al.* (2010), Zhang *et al.* (2010), Fremdt *et al.* (2012), Horváth & Kokoszka (2012), Kraus & Panaretos (2012), Horváth *et al.* (2013), Fremdt *et al.* (2013) and Boente *et al.* (2014).

For instance, Benko *et al.* (2009) and Horváth & Kokoszka (2012, Chapter 5) have developed asymptotic functional testing procedures for the equality of two mean functions. For the more involved problem of testing the equality of covariance functions, Panaretos *et al.* (2010) and Fremdt *et al.* (2012) have developed corresponding testing procedures in the two-sample problem. Critical points of these testing procedures are typically obtained using asymptotic approximations of the distributions of the test statistics used under validity of the null hypothesis. In this context, the main tools utilized are the functional principal components (FPC's) and the associated Karhunen-Loève expansion (KLE); see, e.g., Reiss & Ogden (2007), Gervini (2008), Yao & Müller (2010), Gabrys *et al.* (2010) and Fremdt *et al.* (2013).

For testing the equality of two covariance functions, Panaretos *et al.* (2010) have derived a functional testing procedure under the assumption of Gaussianity, while Fremdt *et al.* (2013) have extended such a functional testing procedure to the non-Gaussian case. Clearly, and due to the complicated statistical functionals involved, the efficacy of these functional testing procedures heavily rely on the accuracy of the obtained asymptotic approximations of the distributions of the test statistics considered under the null hypothesis. Simulation studies, however, suggest that the quality of some asymptotic approximations is questionable. This is not only true for small or moderate sample sizes, that are of paramount importance in practical applications, but also in situations where the assumptions under which the asymptotic results have been derived (e.g., Gaussianity) are not satisfied in practice; see, e.g., Fremdt *et al.* (2013, Section 4).

To improve such asymptotic approximations, bootstrap-based testing inference for functional data have been considered by some authors in the literature. For instance, Benko *et al.* (2009) have considered, among other things, testing the equality of mean functions in the two-sample problem and have applied a bootstrap procedure to obtain critical values of the test statistics used. This bootstrap procedure resamples the original set of functional observations themselves without imposing the null hypothesis and, therefore, its validity rely on the particular test statistic used. That is, the bootstrap approach used does not generate functional pseudo-observations that satisfy the null hypothesis of interest. Therefore, it is not clear if this procedure can be applied to other test statistics, to different testing problems or to the case where more than two populations of functional observations are compared. Similarly, and for the case of comparing the mean functions of two populations of functional observations, Zhang *et al.* (2010) have considered a bootstrap procedure that generates functional pseudo-observations which do not satisfy the null hypothesis. Thus, the validity of this approach depends on the specific test statistic used. A different idea for improving asymptotic approximations

has been used by Boente *et al.* (2014) in the context of testing the equality of several covariance functions, by applying a bootstrap procedure in order to calibrate the critical values of the test statistic used. Again, this bootstrap approach is tailor made for the particular test statistic considered and does involve any resampling of the functional observations themselves. Finally, permutation tests for equality of covariance operators applied to different distance measures between two covariance functions have been considered by Pigoli *et al.* (2014).

We investigate properties of an alternative and general bootstrap-based testing methodology for functional data, which is potentially applicable for different testing problems, different test statistics and for more than two populations. Among other things, the bootstrap-based procedure proposed can be applied to the important problem of comparing the mean functions or the covariance functions between several populations. The basic idea behind this testing methodology is to bootstrap the observed functional data set in such a way that the obtained functional pseudo-observations satisfy the null hypothesis of interest. This requirement leads to a particular bootstrap scheme that automatically generates pseudo-functional observations with identical mean functions (when testing the equality of mean functions) or identical covariance functions (when testing the equality of covariance functions) among the different populations. This common mean function is the estimated pooled mean function (when testing the equality of mean functions) and the common covariance function is the estimated pooled covariance function (when testing the equality of covariance functions) of the observed functional data. A given test statistic of interest is then calculated using the bootstrap functional pseudo-observations and its distribution is evaluated by means of Monte Carlo simulations. As an example, we show that this bootstrap-based functional testing procedure consistently estimates the distribution of the test statistics under the null hypothesis, proposed by Benko *et al.* (2009) and Horváth & Kokoszka (2012, Chapter 5), for the problem of testing the equality of two mean functions, and by Panaretos *et al.* (2011), Fremdt *et al.* (2013) and Boente *et al.* (2014), for the problem of testing the equality of two covariance functions. The theoretically established asymptotic validity of the suggested bootstrap-based functional testing methodology is further gauged by extensive simulations and coincides with accurate approximations of the distributions of interest in finite sample situations. These accurate approximations lead to a very good size and power behavior of the test statistics considered.

The paper is organized as follows. In Section 2, we first present the suggested bootstrap-based methodology applied to the problem of testing equality of the covariance functions in the functional set-up. We then extend the discussion to the problem of testing equality of the mean functions between several groups. Some pertinent observations through remarks on the testing methodology considered are also included. In Section 3, we provide theoretical results which justify the validity of the suggested bootstrap-based testing methodology applied to some test statistics recently considered in the literature. In Section 4, we evaluate the finite sample behavior of the proposed bootstrap-based testing procedures by means of several simulations and compare the results obtained with those based on classical asymptotic approximations. An application to a real-life dataset is also presented. Some concluding remarks are made in Section 5. Finally, auxiliary results and proofs of the main results

are compiled in the Appendix.

2 BOOTSTRAP-BASED FUNCTIONAL TESTING METHODOLOGY

2.1 MODEL AND ASSUMPTIONS

We work with functional data in the form of random functions $X := X(t) := X(\omega, t)$, defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with values in the separable Hilbert-space $\mathcal{H} = L^2(\mathcal{I})$, the space of squared-integrable \mathbb{R} -valued functions on the compact interval $\mathcal{I} = [0, 1]$. We denote by $\mu(t) := \mathbb{E}[X(t)]$, (for almost all) $t \in \mathcal{I}$, the mean function of X , i.e., the unique function $\mu \in L^2(\mathcal{I})$ such that $\mathbb{E} \langle X, x \rangle = \langle \mu, x \rangle$, $x \in L^2(\mathcal{I})$. We also denote by $C(t, s) := \text{Cov}[X(t), X(s)] := \mathbb{E}[(X(t) - \mu(t))(X(s) - \mu(s))]$, $t, s \in \mathcal{I}$, the covariance function (kernel) of X , and by $\mathcal{C}(f) = \mathbb{E}[\langle X - \mu, f \rangle (X - \mu)]$, for $f \in L^2(\mathcal{I})$, the covariance operator of X . It is easily seen that $\mathcal{C}(f)(t) = \int_{\mathcal{I}} C(t, s) f(s) ds$, i.e., \mathcal{C} is an integral operator with kernel C ; note that C is a Hilbert-Schmidt operator provided that $\mathbb{E} \|X\|^2 < \infty$. Throughout the paper we assume that $\langle f, g \rangle = \int_{\mathcal{I}} f(t) g(t) dt$, $\|f\|^2 = \langle f, f \rangle$, and that all functions considered are elements of the separable Hilbert-space $L^2(\mathcal{I})$. Finally, the operator $u \otimes v : L^2 \mapsto L^2$ is defined as $(u \otimes v)w = \langle v, w \rangle u$, $u, v \in L^2$, and we denote by $\|\mathcal{C}\|_S$ the Hilbert-Schmidt norm of the covariance operator \mathcal{C} .

Throughout the paper, it is also assumed that we have available a collection of random functions satisfying

$$X_{i,j}(t) = \mu_i(t) + \epsilon_{i,j}(t), \quad i = 1, 2, \dots, K, \quad j = 1, 2, \dots, n_i, \quad t \in \mathcal{I}, \quad (1)$$

where K ($2 \leq K < \infty$) denotes the number of populations (groups), n_i denotes the number of observations in the i -th population and $N = \sum_{i=1}^K n_i$ denotes the total number of observations. We also assume that the K populations are independent and, for each $i \in \{1, 2, \dots, K\}$ and $j = 1, 2, \dots, n_i$, the $\epsilon_{i,j}$ are independent and identical distributed random elements with $\mathbb{E}[\epsilon_{i,j}(t)] = 0$, $t \in \mathcal{I}$, and $\mathbb{E} \|\epsilon_{i,j}\|^4 < \infty$.

Denote by (λ_k, φ_k) , $k = 1, 2, \dots$, the eigenvalues/eigenfunctions of the covariance operator \mathcal{C} , i.e.

$$\lambda_k \varphi_k(t) = \mathcal{C}(\varphi_k)(t) := \int_{\mathcal{I}} C(t, s) \varphi_k(s) ds, \quad t \in \mathcal{I}, \quad k = 1, 2, \dots$$

Throughout the paper it is assumed that $\lambda_1 > \lambda_2 > \dots > \lambda_p > \lambda_{p+1}$, i.e., there exists at least p distinct (positive) eigenvalues of the covariance operator \mathcal{C} .

2.2 TESTING THE EQUALITY OF COVARIANCE FUNCTIONS

In this section, we describe the suggested bootstrap-based functional testing methodology for testing the equality of covariance functions (resp. covariance operators) for a (finite) number of populations. Since testing the equality of covariance functions is equivalent to testing the equality of covariance operators, as in Fremdt *et al.* (2012), we confine our attention to the former test.

Let $\mathbf{X}_N = \{X_{i,j}(t), i \in \{1, 2, \dots, K\}, j = 1, 2, \dots, n_i, t \in \mathcal{I}\}$ be the observed collection of random functions satisfying (1). Let $C_i(t, s)$, $t, s \in \mathcal{I}$, be the covariance functions in the i -th population, i.e., for each $i \in \{1, 2, \dots, K\}$, $C_i(t, s) := \text{Cov}[X_{i,j}(t), X_{i,j}(s)] := \mathbb{E}[(X_{i,j}(t) - \mu_i(t))(X_{i,j}(s) - \mu_i(s))]$, where $\mu_i(t) := \mathbb{E}[X_{i,j}(t)]$, $j = 1, 2, \dots, n_i$. Our aim is to test the null hypothesis

$$H_0 : C_1 = C_2 = \dots = C_K \quad (2)$$

versus the alternative hypothesis

$$H_1 : \exists (k, l) \in \{1, 2, \dots, K\} \text{ with } k \neq l \text{ such that } C_k \neq C_l. \quad (3)$$

Notice that the equality in the null hypothesis (2) is in the space $(L^2(\mathcal{I} \times \mathcal{I}), \|\cdot\|)$, i.e., $C_k = C_l$, for any pair of indices $(k, l) \in \{1, 2, \dots, K\}$, with $k \neq l$, means that $\|C_k - C_l\| = 0$, and the alternative hypothesis (3) means that $\|C_k - C_l\| > 0$.

2.2.1 THE BOOTSTRAP-BASED TESTING PROCEDURE

Let T_N be a given test statistic of interest for testing hypothesis (2) which is based on the functional observations \mathbf{X}_N . Assume, without loss of generality, that T_N rejects the null hypothesis H_0 when $T_N > d_{N,\alpha}$, where for $\alpha \in (0, 1)$, $d_{N,\alpha}$ denotes the critical value of this test. The bootstrap-based functional testing procedure for testing hypotheses (2)-(3) can be described as follows:

Step 1: First calculate the sample mean functions in each population

$$\bar{X}_{i,n_i}(t) = \frac{1}{n_i} \sum_{j=1}^{n_i} X_{i,j}(t), \quad t \in \mathcal{I}, \quad i \in \{1, 2, \dots, K\}.$$

Step 2: Calculate the residual functions in each population, i.e., for each $i \in \{1, 2, \dots, K\}$,

$$\hat{\epsilon}_{i,j}(t) = X_{i,j}(t) - \bar{X}_{i,n_i}(t), \quad t \in \mathcal{I}, \quad j = 1, 2, \dots, n_i.$$

Step 3: Generate bootstrap functional pseudo-observations $X_{i,j}^*(t)$, $t \in \mathcal{I}$, $i \in \{1, 2, \dots, K\}$, $j = 1, 2, \dots, n_i$, according to

$$X_{i,j}^*(t) = \bar{X}_{i,n_i}(t) + \epsilon_{i,j}^*(t), \quad t \in \mathcal{I}, \quad (4)$$

where

$$\epsilon_{i,j}^*(t) = \hat{\epsilon}_{I,J}(t), \quad t \in \mathcal{I},$$

and (I, J) is the following pair of random variables. The random variable I takes values in the set $\{1, 2, \dots, K\}$ with probability $P(I = i) = n_i/N$ for $i \in \{1, 2, \dots, K\}$, and, given $I = i$, the random variable J has the discrete uniform distribution in the set $\{1, 2, \dots, n_i\}$, i.e., $P(J = j | I = i) = 1/n_i$ for $i \in \{1, 2, \dots, K\}$, $j = 1, 2, \dots, n_i$.

Step 4: Let T_N^* be the same statistic as T_N but calculated using the bootstrap functional pseudo-observations $X_{i,j}^*$, $i \in \{1, 2, \dots, K\}$; $j = 1, 2, \dots, n_i$. Denote by $D_{N,T}^*$ the distribution function of T_N^* given the functional observations \mathbf{X}_N .

Step 5: For any given $\alpha \in (0, 1)$, reject the null hypothesis H_0 if and only if

$$T_N > d_{N,\alpha}^*,$$

where $d_{N,\alpha}^*$ denotes the α -quantile of $D_{N,T}^*$, i.e., $D_{N,T}^*(d_{N,\alpha}^*) = 1 - \alpha$.

Notice that the distribution $D_{N,T}^*$ can be evaluated by means of Monte-Carlo.

Clearly, and since the random functions $\epsilon_{i,j}^*(t)$ are generated independently from each other, for any two different pairs of indices, say (i_1, j_1) and (i_2, j_2) , the corresponding bootstrap functional pseudo-observations $X_{i_1,j_1}^*(t)$ and $X_{i_2,j_2}^*(t)$ are independent. Furthermore, observe that the random selection of the error function $\epsilon_{i,j}^*(t)$ in Step 4 of the above bootstrap algorithm, is equivalent to selecting $\epsilon_{i,j}^*(t)$ randomly with probability $1/N$ from the entire set of available and estimated residual functions $\{\hat{\epsilon}_{r,s}(t) : r = 1, 2, \dots, K \text{ and } s = 1, 2, \dots, n_r\}$. Hence, conditional on the observed functional data \mathbf{X}_N , the functional pseudo-observations $X_{i,j}^*(t)$ have the following first and second order properties:

$$\mathbb{E}[X_{i,j}^*(t)] = \bar{X}_{i,n_i}(t) + \mathbb{E}[\epsilon_{i,j}^*(t)] = \bar{X}_{i,n_i}(t) + \frac{1}{N} \sum_{i=1}^K \sum_{j=1}^{n_i} \hat{\epsilon}_{i,j}(t) = \bar{X}_{i,n_i}(t), \quad t \in \mathcal{I},$$

since $\sum_{j=1}^{n_i} \hat{\epsilon}_{i,j}(t) = \sum_{j=1}^{n_i} (X_{i,j}(t) - \bar{X}_{i,n_i}(t)) = 0$, within each population $i \in \{1, 2, \dots, K\}$.

Moreover,

$$\begin{aligned} Cov[X_{i,j}^*(t), X_{i,j}^*(s)] &= \mathbb{E}[\epsilon_{i,j}^*(t)\epsilon_{i,j}^*(s)] = \frac{1}{N} \sum_{i=1}^K \sum_{j=1}^{n_i} \hat{\epsilon}_{i,j}(t)\hat{\epsilon}_{i,j}(s) \\ &= \sum_{i=1}^K \frac{n_i}{N} \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{i,j}(t) - \bar{X}_{i,n_i}(t))(X_{i,j}(s) - \bar{X}_{i,n_i}(s)) \\ &= \sum_{i=1}^K \frac{n_i}{N} \hat{C}_{i,n_i}(t, s) = \hat{C}_N(t, s), \quad t, s \in \mathcal{I}, \end{aligned}$$

where

$$\hat{C}_{i,n_i}(t, s) = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{i,j}(t) - \bar{X}_{i,n_i}(t))(X_{i,j}(s) - \bar{X}_{i,n_i}(s)), \quad t, s \in \mathcal{I},$$

is the sample estimator of the covariance function $C_i(t, s)$ for the i -th population and $\hat{C}_N(t, s)$ is the corresponding pooled covariance function estimator.

Thus, and conditional on the observed functional data \mathbf{X}_N , the bootstrap generated functional pseudo-observations $X_{i,j}^*(t)$ have, within each population $i \in \{1, 2, \dots, K\}$, the same mean function $\bar{X}_{i,n_i}(t)$, which may be different for different populations. Furthermore, the covariance function in each population is identical and equal to the pooled sample covariance function $\hat{C}_N(t, s)$. That is, the functional pseudo-observations $X_{i,j}^*(t)$, satisfy the null hypothesis (2). This basic property of the $X_{i,j}^*(t)$'s allows us to use these bootstrap observations to evaluate the distribution of some test statistic T_N of interest under the null hypothesis. This is achieved by using the distribution of T_N^* as an estimator of the distribution of T_N , where T_N^* is the same statistical functional as T_N calculated using the bootstrap functional pseudo-observations $\mathbf{X}_N^* = \{X_{i,j}^*(t), i = 1, 2, \dots, K, j = 1, 2, \dots, n_K, t \in \mathcal{I}\}$.

Since, as we have seen, the set of pseudo-observations used to calculate T_N^* satisfy the null hypothesis, we expect that the distribution of the pseudo random variable T_N^* will mimic correctly the distribution of T_N under the null. In the next section we show that this is indeed true for two particular test statistics proposed in the literature. However, since our bootstrap methodology is not designed or tailor made for any particular test statistic, its range of validity is not restricted to these two particular test statistics.

2.3 TESTING THE EQUALITY OF MEAN FUNCTIONS

We assume again that we have available a collection of curves \mathbf{X}_N , satisfying (1). Recall that $\mu_i(t)$, $t \in \mathcal{I}$ denote the mean functions of the curves in the i -th population, i.e., for each $i \in \{1, 2, \dots, K\}$, $\mu_i(t) := \mathbb{E}[X_{i,j}(t)]$, $j = 1, 2, \dots, n_i$.

The basic idea used in the bootstrap resampling algorithm of Section 2.2.1 enables its adaption/modification to deal with different testing problems related to the comparison of K populations of functional observations. For instance, suppose that we are interested in testing the null hypothesis that the K populations have identical mean functions, i.e.,

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_K \quad (5)$$

versus the alternative hypothesis

$$H_1 : \exists (k, l) \in \{1, 2, \dots, K\} \text{ with } k \neq l \text{ such that } \mu_k \neq \mu_l. \quad (6)$$

As in the previous section, equality in the null hypothesis (5) is in the space $(L^2(\mathcal{I}), \|\cdot\|)$, i.e., $\mu_k = \mu_l$, for any pair of indices $(k, l) \in \{1, 2, \dots, K\}$, with $k \neq l$, means that $\|\mu_k - \mu_l\| = 0$, and the alternative hypothesis (6) means that $\|\mu_k - \mu_l\| > 0$.

Such a testing problem can be easily addressed by changing appropriately Step 3 of the bootstrap resampling algorithm of Section 2.2.1. In particular, we replace equation (4) in Step 3 of this algorithm by the following equation

$$X_{i,j}^+(t) = \bar{X}_N(t) + \epsilon_{i,j}^+(t), \quad t \in \mathcal{I}, \quad (7)$$

where

$$\bar{X}_N(t) = \frac{1}{N} \sum_{i=1}^K \sum_{j=1}^{n_i} X_{i,j}(t), \quad t \in \mathcal{I},$$

is the pooled mean function estimator and $\epsilon_{i,j}^+(t) = \hat{\epsilon}_{i,J}(t)$, $t \in \mathcal{I}$, where J is a discrete random variable with $P(J = j) = 1/n_i$ for every $j = 1, 2, \dots, n_i$, $i \in \{1, 2, \dots, K\}$. Thus, the bootstrap error functions $\epsilon_{i,j}^+(t)$ for population i appearing in equation (7) are generated by randomly selecting a residual function from the set of estimated residual functions $\hat{\epsilon}_{i,j}(t)$ belonging to the same population $i \in \{1, 2, \dots, K\}$. This ensures that the covariance structure of the functional observations in each

population is retained by the bootstrap resampling algorithm, which may be different for different populations, despite the fact that the bootstrap procedure generates K populations of independent bootstrap functional pseudo-observations that have the same mean function. In particular, conditional on the observed functional data \mathbf{X}_N , we have for the bootstrap functional pseudo-observations $X_{i,j}^+(t)$, that,

$$\mathbb{E}[X_{i,j}^+(t)] = \bar{X}_N(t), \quad t \in \mathcal{I},$$

and

$$\begin{aligned} \text{Cov}[X_{i,j}^+(t), X_{i,j}^+(s)] &= \mathbb{E}[\epsilon_{i,j}^+(t)\epsilon_{i,j}^+(s)] = \frac{1}{n_i} \sum_{j=1}^{n_i} \hat{\epsilon}_{i,j}(t)\hat{\epsilon}_{i,j}(s) \\ &= \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{i,j}(t) - \bar{X}_{i,n_i}(t))(X_{i,j}(s) - \bar{X}_{i,n_i}(s)) \\ &= \hat{C}_{i,n_i}(t, s), \quad t, s \in \mathcal{I}. \end{aligned}$$

Remark 2.1 Notice that if we use $X_{i,j}^\circ(t) = \bar{X}_N(t) + \epsilon_{i,j}^*(t)$, $t \in \mathcal{I}$, to generate the bootstrap pseudo-observations instead of equation (7) with $\epsilon_{i,j}^*(t)$ defined as in Step 3 of the algorithm in Section 2.2.1, then the $X_{i,j}^\circ(t)$ will have in the K groups an identical mean function equal to $\bar{X}_N(t)$ and an identical covariance function equal to $\hat{C}_N(t, s)$. This may be of particular interest if one is interested in testing *simultaneously* the equality of mean functions and covariance functions between the K populations.

Remark 2.2 If distributional assumptions have been imposed on the observed random functions $X_{i,j}(t)$, then efficiency considerations may suggest that such assumptions should be also taken into account in the implementation of the bootstrap resampling algorithms which are used to generate the bootstrap functional pseudo-observations. For instance, the assumption of Gaussianity of the random paths $X_{i,j}(t)$, $t \in \mathcal{I}$, can be incorporated in our bootstrap testing algorithm by allowing for the functional bootstrap pseudo-observations to follow a Gaussian processes on \mathcal{I} with a mean and covariance function specified according the null hypothesis of interest.

3 BOOTSTRAP VALIDITY

In this section, we establish the validity of the introduced bootstrap-based functional testing methodology applied to some test statistics recently proposed in the literature for the important problems of testing the equality of mean functions or covariance functions between two populations.

3.1 Testing the equality of two covariance functions

3.1.1 Test Statistics and Limiting Distributions

For testing the equality of two covariance operators, it is natural to evaluate the Hilbert-Schmidt norm of the difference of the corresponding sample covariance operators $\widehat{\mathcal{C}}_1$ and $\widehat{\mathcal{C}}_2$, defined as

$$\widehat{\mathcal{C}}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{i,j} - \bar{X}_{i,n_i}) \otimes (X_{i,j} - \bar{X}_{i,n_i}), \quad i = 1, 2,$$

or, equivalently, as

$$\widehat{\mathcal{C}}_i(f)(t) = \frac{1}{n_i} \sum_{j=1}^{n_i} \langle X_{i,j} - \bar{X}_{i,n_i}, f \rangle (X_{i,j}(t) - \bar{X}_{i,n_i}(t)), \quad f \in L^2(\mathcal{I}), \quad t \in \mathcal{I}, \quad i = 1, 2.$$

Such an approach has been recently proposed by Boente *et al.* (2014) by considering the test statistic

$$T_N = N \|\widehat{\mathcal{C}}_1 - \widehat{\mathcal{C}}_2\|_S^2,$$

where $N = n_1 + n_2$. If $n_1/N \rightarrow \theta_1 \in (0, 1)$, $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and the null hypothesis H_0 given in (2) with $K = 2$ is true, then, as $n_1, n_2 \rightarrow \infty$, Boente *et al.* (2014) showed that T_N converges weakly to $\sum_{l=1}^{\infty} \lambda_l Z_l^2$, where Z_l are independently distributed standard Gaussian random variables and λ_l are the eigenvalues of the pooled operator $\mathcal{B} = \theta_1^{-1} \mathcal{B}_1 + (1 - \theta_1)^{-1} \mathcal{B}_2$, where \mathcal{B}_i is the covariance operator of the limiting Gaussian random element U_i to which $\sqrt{n_i}(\widehat{\mathcal{C}}_i - \mathcal{C}_i)$ converges weakly as $n_i \rightarrow \infty$, $i = 1, 2$. Since the limiting distribution of T_N depends on the unknown infinite eigenvalues λ_l , $l \geq 1$, implementation of this asymptotic result for calculating critical values of the test is difficult. With this in mind, Boente *et al.* (2014) have proposed a bootstrap calibration procedure of the distribution of the test statistic T_N .

Another, related, approach for testing the equality of two covariance functions, is to evaluate the distance between the sample covariance functions $\widehat{C}_{1,n_1}(t, s)$ and $\widehat{C}_{2,n_2}(t, s)$, $t, s \in \mathcal{I}$, of each group and the pooled sample covariance function $\widehat{C}_N(t, s)$, $t, s \in \mathcal{I}$, based on the entire set of functional observations. Therefore, looking at projections of $\widehat{C}_{1,n_1}(t, s) - \widehat{C}_{2,n_2}(t, s)$, $t, s \in \mathcal{I}$, on certain directions reduces the dimensionality of the problem. Such approaches have been considered by Panaretos *et al.* (2010) (for Gaussian curves) and Fremdt *et al.* (2012), (for non-Gaussian curves), where the asymptotic distributions of the corresponding test statistics proposed under the null hypothesis have been derived

More specifically, denote by $(\widehat{\lambda}_k, \widehat{\varphi}_k)$, $k = 1, 2, \dots, N$, the eigenvalues/eigenfunctions of the pooled sample covariance operator $\widehat{\mathcal{C}}_N$ defined by the kernel $\widehat{C}_N(t, s)$, i.e.,

$$\widehat{\lambda}_k \widehat{\varphi}_k(t) = \widehat{\mathcal{C}}_N(\widehat{\varphi}_k)(t) = \int_0^1 \widehat{C}_N(t, s) \widehat{\varphi}_k(s) ds, \quad t \in \mathcal{I}, \quad k = 1, 2, \dots, N,$$

with $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \dots$. (We can and will assume that the $\hat{\varphi}_k(t)$, $k = 1, 2, \dots, N$, $t \in \mathcal{I}$, form an orthonormal system.) Select a natural number p and consider, for $i = 1, 2, \dots, p$, the projections

$$\hat{a}_{k,j}(i) = \langle X_{k,j} - \bar{X}_{k,n_k}, \hat{\varphi}_i \rangle = \int_{\mathcal{I}} (X_{k,j}(t) - \bar{X}_{k,n_k}(t)) \hat{\varphi}_i(t) dt, \quad j = 1, 2, \dots, n_k, \quad k = 1, 2.$$

For $1 \leq r, m \leq p$, consider the matrices \hat{A}_{k,n_k} , $k = 1, 2$, with elements

$$\hat{A}_{k,n_k}(r, m) = \frac{1}{n_k} \sum_{j=1}^{n_k} \hat{a}_{k,j}(r) \hat{a}_{k,j}(m), \quad k = 1, 2.$$

Notice that, for $1 \leq r, m \leq p$, $\hat{\Delta}_N(r, m) := \hat{A}_{1,n_1}(r, m) - \hat{A}_{2,n_2}(r, m)$ is the projection of the difference $\hat{C}_{1,n_1}(t, s) - \hat{C}_{2,n_2}(t, s)$ in the direction of $\hat{\varphi}_r(t) \hat{\varphi}_m(s)$, $t, s \in \mathcal{I}$.

Panaretos *et al.* (2010) considered then the test statistic

$$T_{p,N}^{(G)} = \frac{n_1 n_2}{N} \sum_{1 \leq r, m \leq p} \frac{\hat{\Delta}_N^2(r, m)}{2 \hat{\lambda}_r \hat{\lambda}_m}.$$

They showed, that, under the assumption that $X_{i,1}(t)$, $i \in \{1, 2\}$, $t \in \mathcal{I}$, are Gaussian processes, and if $n_1, n_2 \rightarrow \infty$ such that $n_1/N \rightarrow \theta_1 \in (0, 1)$, $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$ and the null hypothesis H_0 given in (2) with $K = 2$ is true, then, $T_{p,N}^{(G)}$ converges weakly to a $\chi_{p(p+1)/2}^2$ distribution.

The non-Gaussian case has been recently investigated by Fremdt *et al.* (2012). In particular, they considered the matrix $\hat{\Delta}_N = (\hat{\Delta}_N(r, m))_{r,m=1,2,\dots,p}$ and defined $\hat{\xi}_N = \text{vech}(\hat{\Delta}_N)$, i.e., the vector containing the elements on and below the main diagonal of $\hat{\Delta}_N$. Fremdt *et al.* (2012) proposed then the test statistic

$$T_{p,N} = \frac{n_1 n_2}{N} \hat{\xi}_N^T \hat{L}_N^{-1} \hat{\xi}_N,$$

where \hat{L}_N is an estimator of the (asymptotic) covariance matrix of $\hat{\xi}_N$. They showed that if $n_1, n_2 \rightarrow \infty$ such that $n_1/N \rightarrow \theta_1 \in (0, 1)$, $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and the null hypothesis H_0 given in (2) with $K = 2$ is true, then, $T_{2,N}$ converges weakly to a $\chi_{p(p+1)/2}^2$ distribution. Furthermore, under the same set of assumptions, consistency of the test $T_{p,N}$ has been established under the alternative, that is when the covariance functions C_1 and C_2 differ.

3.1.2 Consistency of the Bootstrap

We apply the bootstrap procedure introduced in Section 2.2.1 to approximate the distributions of the test statistics T_N , $T_{p,N}^{(G)}$ and of $T_{p,N}$ under the null hypothesis. To this end, let $X_{i,j}^*(t)$, $i \in \{1, 2\}$, $j = 1, 2, \dots, n_i$, $t \in \mathcal{I}$, be the bootstrap functional pseudo-observations generated according to this bootstrap procedure. Let

$$T_N^* = N \|\hat{\mathcal{C}}_1^* - \hat{\mathcal{C}}_2^*\|_S^2,$$

where \hat{C}_1^* and \hat{C}_2^* are the sample covariance operators of the two groups but calculated using the bootstrap functional pseudo-observations $X_{i,j}^*(t)$, $i \in \{1, 2\}$, $j = 1, 2, \dots, n_i$, $t \in \mathcal{I}$. Let

$$T_{p,N}^{*(G)} = \frac{n_1 n_2}{N} \sum_{1 \leq r, m \leq p} \frac{1}{2} \frac{\hat{\Delta}_N^{*2}(r, m)}{\hat{\lambda}_r^* \hat{\lambda}_m^*},$$

where $\hat{\Delta}_N^{*2}(r, m)$ and $\hat{\lambda}_r^*$ are the same statistics as $\hat{\Delta}_N^2(r, m)$ and $\hat{\lambda}_r$ appearing in $T_{p,N}^{(G)}$ but calculated using the bootstrap functional pseudo-observations $X_{i,j}^*(t)$, $i \in \{1, 2\}$, $j = 1, 2, \dots, n_i$, $t \in \mathcal{I}$. Similarly, let

$$T_{p,N}^* = \frac{n_1 n_2}{N} \hat{\xi}_N^{*T} \hat{L}_N^{*-1} \hat{\xi}_N^*,$$

where $\hat{\xi}_N^*$ and \hat{L}_N^* are the same statistics as $\hat{\xi}_N$ and \hat{L}_N appearing in $T_{p,N}$ but calculated using the bootstrap functional pseudo-observations $X_{i,j}^*(t)$, $i \in \{1, 2\}$, $j = 1, 2, \dots, n_i$, $t \in \mathcal{I}$. The following results are then true.

Theorem 3.1 *If $E\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and $n_1/N \rightarrow \theta_1 \in (0, 1)$, then, as $n_1, n_2 \rightarrow \infty$,*

$$\sup_{x \in \mathfrak{R}} \left| \mathbb{P}(T_N^* \leq x \mid \mathbf{X}_N) - \mathbb{P}_{H_0}(T_N \leq x) \right| \rightarrow 0, \quad \text{in probability,}$$

where $\mathbb{P}_{H_0}(T_N \leq x)$, $x \in \mathfrak{R}$, denotes the distribution function of T_N when H_0 given in (2) with $K = 2$ is true and $\mathcal{B}_1 = \mathcal{B}_2$.

Notice that by the above theorem, the suggested bootstrap procedure leads to consistent estimation of the critical values of the test T_N for which the asymptotic approximation discussed in Section 3.1.1 is difficult to implement in practice.

Theorem 3.2 *Assume that $X_{i,1}(t)$, $i \in \{1, 2\}$, $t \in \mathcal{I}$, are Gaussian processes. If $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and $n_1/N \rightarrow \theta_1 \in (0, 1)$, then, as $n_1, n_2 \rightarrow \infty$,*

$$\sup_{x \in \mathfrak{R}} \left| \mathbb{P}(T_{p,N}^{*(G)} \leq x \mid \mathbf{X}_N) - \mathbb{P}_{H_0}(T_{p,N}^{(G)} \leq x) \right| \rightarrow 0, \quad \text{in probability,}$$

where $\mathbb{P}_{H_0}(T_{p,N}^{(G)} \leq x)$, $x \in \mathfrak{R}$, denotes the distribution function of $T_{p,N}^{(G)}$ when H_0 given in (2) with $K = 2$ is true.

Theorem 3.3 *If $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and $n_1/N \rightarrow \theta_1 \in (0, 1)$, then, as $n_1, n_2 \rightarrow \infty$,*

$$\sup_{x \in \mathfrak{R}} \left| \mathbb{P}(T_{p,N}^* \leq x \mid \mathbf{X}_N) - \mathbb{P}_{H_0}(T_{p,N} \leq x) \right| \rightarrow 0, \quad \text{in probability,}$$

where $\mathbb{P}_{H_0}(T_{p,N} \leq x)$, $x \in \mathfrak{R}$, denotes the distribution function of $T_{p,N}$ when H_0 given in (2) with $K = 2$ is true.

Remark 3.1 Notice that if H_1 is true and $\|\mathcal{C}_1 - \mathcal{C}_2\|_S > 0$, then, we have that, as $n_1, n_2 \rightarrow \infty$, $T_N \rightarrow \infty$, in probability. Theorem 3.1 implies then that the test T_N based on the bootstrap critical values obtained using the distribution of the test T_N^* is consistent, i.e., its power approaches unity, as $n_1, n_2 \rightarrow \infty$. Furthermore, if H_1 is true and if $\xi = \text{vech}(D) \neq 0$, where D is the $p \times p$ matrix $D = ((\int_{\mathcal{I}} \int_{\mathcal{I}} (C_1(t, s) - C_2(t, s)) \varphi_i(t) \varphi_j(s) dt ds, 1 \leq i, j \leq p)$, then, under the same assumptions as in Theorem 3 of Fremdt *et al.* (2012), we have that, as $n_1, n_2 \rightarrow \infty$, $T_{p,N} \rightarrow \infty$, in probability. Theorem 3.3 implies then that, under these assumptions, the test $T_{p,N}$ based on the bootstrap critical values obtained using the distribution of the test $T_{p,N}^*$ is also consistent, i.e., its power approaches unity, as $n_1, n_2 \rightarrow \infty$.

3.2 Testing the equality of two mean functions

3.2.1 Test Statistics and Limiting Distributions

For testing the equality of two mean functions, it is natural to compute the $L^2(\mathcal{I})$ -distance between the two sample mean functions $\bar{X}_{1,n_1}(t)$ and $\bar{X}_{2,n_2}(t)$, $t \in \mathcal{I}$. Such an approach was considered by Benko *et al.* (2009) and Horváth & Kokoszka (2012, Chapter 5) using the test statistic

$$S_N = \frac{n_1 n_2}{N} \|\bar{X}_{1,n_1} - \bar{X}_{2,n_2}\|^2.$$

If $n_1/N \rightarrow \theta_1 \in (0, 1)$, $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and the null hypothesis H_0 given in (5) with $K = 2$ is true, then, as $n_1, n_2 \rightarrow \infty$, they showed that S_N converges weakly to $\int_{\mathcal{I}} \Gamma^2(t) dt$, where $\{\Gamma(t) : t \in \mathcal{I}\}$ is a Gaussian process satisfying $\mathbb{E}[\Gamma(t)] = 0$ and $\mathbb{E}[\Gamma(t)\Gamma(s)] = (1 - \theta) C_1(t, s) + \theta C_2(t, s)$, $t, s \in \mathcal{I}$. They have also showed consistency of the test, in the sense that if the alternative hypothesis H_1 given in (6) is true, then, as $n_1, n_2 \rightarrow \infty$, $S_N \rightarrow \infty$, in probability.

Notice that the limiting distribution of the test statistic S_N depends on the unknown covariance functions C_1 and C_2 . Hence, analytical calculation of critical values of this test turns out to be difficult in practice. To overcome this problem, Horváth & Kokoszka (2012, Chapter 5), considered two projections versions of the test statistic S_N . Note that, using the KLE, it follows that $\Gamma(t) = \sum_{k=1}^{\infty} \sqrt{\tau_k} N_k \phi_k(t)$ and $\int_{\mathcal{I}} \Gamma^2(t) dt = \sum_{k=1}^{\infty} \tau_k N_k^2$, $t \in \mathcal{I}$, where N_k , $k = 1, 2, \dots$, is a sequence of independent standard Gaussian random variables, and $\tau_1 \geq \tau_2 \geq \dots$ and $\phi_1(t), \phi_2(t), \dots$, $t \in \mathcal{I}$, are the eigenvalues and eigenfunctions of the operator \mathcal{Z} determined by the kernel $Z(t, s) = (1 - \theta) C_1(t, s) + \theta C_2(t, s)$, $t, s \in \mathcal{I}$, $\theta \in (0, 1)$. In view of this, Horváth & Kokoszka (2012, Chapter 5) considered projections on the space determined by the p leading eigenfunctions of the operator \mathcal{Z} . Assume that the eigenvalues of the operator \mathcal{Z} satisfy $\tau_1 > \tau_2 > \dots > \tau_p > \tau_{p+1}$, i.e., that there exists at least p distinct (positive) eigenvalues of the operator \mathcal{Z} . Let

$$\hat{a}_i = \langle \bar{X}_{1,n_1} - \bar{X}_{2,n_2}, \hat{\phi}_i \rangle = \int_{\mathcal{I}} (\bar{X}_{1,n_1}(t) - \bar{X}_{2,n_2}(t)) \hat{\phi}_i(t) dt, \quad i = 1, 2, \dots, p,$$

be the projection of the difference $\overline{X}_{1,n_1}(t) - \overline{X}_{2,n_2}(t)$, $t \in \mathcal{I}$, into the linear space spanned by $\hat{\phi}_1(t)$, $\hat{\phi}_2(t), \dots, \hat{\phi}_p(t)$, $t \in \mathcal{I}$, the eigenfunctions related to the sample estimator $\hat{Z}_N(t, s)$ of the kernel $Z(t, s)$, $t, s \in \mathcal{I}$. Based on the above, Horváth & Kokoszka (2012, Chapter 5) considered the following test statistics

$$S_{p,N}^{(1)} = \frac{n_1 n_2}{N} \sum_{k=1}^p \frac{\hat{a}_k^2}{\hat{\tau}_k} \quad \text{and} \quad S_{p,N}^{(2)} = \frac{n_1 n_2}{N} \sum_{k=1}^p \hat{a}_k^2.$$

If $n_1/N \rightarrow \theta_1 \in (0, 1)$, $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and the null hypothesis H_0 given in (5) with $K = 2$ is true, then, as $n_1, n_2 \rightarrow \infty$, they showed that, $S_{p,N}^{(1)}$ converges weakly to a χ_p^2 -distribution while $S_{p,N}^{(2)}$ converges weakly to $\sum_{k=1}^p \tau_k N_k^2$. Under the assumption that $\mu_1(t) - \mu_2(t)$, $t \in \mathcal{I}$, is not orthogonal to the linear span of $\phi_1(t), \phi_2(t), \dots, \phi_p(t)$, $t \in \mathcal{I}$, they have also showed consistency, in the sense that, if $n_1/N \rightarrow \theta_1 \in (0, 1)$, $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and the alternative hypothesis H_1 given in (6) with $K = 2$ is true, then, as $n_1, n_2 \rightarrow \infty$, $S_{p,N}^{(1)} \rightarrow \infty$ and $S_{p,N}^{(2)} \rightarrow \infty$, in probability.

3.2.2 Consistency of the Bootstrap

To approximate the distribution of the test statistics S_N , $S_{p,N}^{(1)}$ and $S_{p,N}^{(2)}$, we apply the bootstrap procedure proposed in Section 2.3. To this end, let $X_{i,j}^+(t)$, $i \in \{1, 2\}$, $j = 1, 2, \dots, n_i$, $t \in \mathcal{I}$, be the bootstrap functional pseudo-observations generated according to this bootstrap algorithm and define

$$\begin{aligned} S_N^+ &= \frac{n_1 n_2}{N} \|\overline{X}_{1,n_1}^+ - \overline{X}_{2,n_2}^+\|^2, \\ S_{p,N}^{+(1)} &= \frac{n_1 n_2}{N} \sum_{k=1}^p \frac{\hat{a}_k^{+2}}{\hat{\tau}_k^+}, \quad \text{and} \\ S_{p,N}^{+(2)} &= \frac{n_1 n_2}{N} \sum_{k=1}^p \hat{a}_k^{+2} \end{aligned}$$

be the same statistic as S_N , $S_{p,N}^{(1)}$ and $S_{p,N}^{(2)}$, respectively, but calculated using the bootstrap functional pseudo-observations $X_{i,j}^+(t)$, $i \in \{1, 2\}$, $j = 1, 2, \dots, n_i$, $t \in \mathcal{I}$. The following results are then true.

Theorem 3.4 *If $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and $n_1/N \rightarrow \theta_1 \in (0, 1)$, then, as $n_1, n_2 \rightarrow \infty$,*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}(S_N^+ \leq x \mid \mathbf{X}_N) - \mathbb{P}_{H_0}(S_N \leq x) \right| \rightarrow 0, \quad \text{in probability,}$$

where $\mathbb{P}_{H_0}(S_N \leq x)$, $x \in \mathbb{R}$, denotes the distribution function of S_N when H_0 given in (5) with $K = 2$ is true.

Notice that by the above theorem, the suggested bootstrap procedure leads to consistent estimation of the critical values of the test S_N for which the asymptotic approximations discussed in Section 3.2.1 is difficult to implement in practice.

Theorem 3.5 *If $\mathbb{E}\|X_{i,1}\|^4 < \infty$, $i \in \{1, 2\}$, and $n_1/N \rightarrow \theta_1 \in (0, 1)$, then, as $n_1, n_2 \rightarrow \infty$,*

$$(i) \quad \sup_{x \in \mathbb{R}} \left| \mathbb{P}(S_{p,N}^{+(1)} \leq x \mid \mathbf{X}_N) - \mathbb{P}_{H_0}(S_{p,N}^{(1)} \leq x) \right| \rightarrow 0, \quad \text{in probability,}$$

and

$$(ii) \quad \sup_{x \in \mathbb{R}} \left| \mathbb{P}(S_{p,N}^{+(2)} \leq x \mid \mathbf{X}_N) - \mathbb{P}_{H_0}(S_{p,N}^{(2)} \leq x) \right| \rightarrow 0, \quad \text{in probability,}$$

where $\mathbb{P}_{H_0}(S_{p,N}^{(1)} \leq x)$ and $\mathbb{P}_{H_0}(S_{p,N}^{(2)} \leq x)$, $x \in \mathbb{R}$, denote the distribution functions of $S_{p,N}^{(1)}$ and $S_{p,N}^{(2)}$, respectively, when H_0 given in (5) with $K = 2$ is true.

Remark 3.2 Under the same assumptions as in Theorem 5.2 of Horváth & Kokoszka (2012) and if $\|\mu_1 - \mu_2\| > 0$ we have that $S_N \rightarrow \infty$ in probability. This result together with Theorem 3.4 imply consistency of the test S_N using the bootstrap critical values of the distribution of the test S_N^+ , i.e., its power approaches unity, as $n_1, n_2 \rightarrow \infty$. Furthermore, if the difference $\mu_1 - \mu_2$ is not orthogonal to the linear span of $\varphi_1, \varphi_2, \dots, \varphi_p$, then, as $n_1, n_2 \rightarrow \infty$, $S_{p,N}^{(1)} \rightarrow \infty$ and $S_{p,N}^{(2)} \rightarrow \infty$, in probability. Theorem 3.5 implies then that, under these conditions, the tests $S_{p,N}^{(1)}$ and $S_{p,N}^{(2)}$ based on the bootstrap critical values of the distributions of the tests $S_{p,N}^{+(1)}$ and $S_{p,N}^{+(2)}$, respectively, are also consistent, i.e., their power approaches unity, as $n_1, n_2 \rightarrow \infty$.

4 NUMERICAL RESULTS

In this section, we evaluate the finite sample behavior of the proposed bootstrap-based functional testing procedures, for testing the equality of two covariance functions or the equality of two mean functions, by means of several simulations and compare our results with those based on classical asymptotic approximations of the distribution of the test statistic. An illustration to an interesting real-life dataset is also presented.

4.1 SIMULATIONS

Following Fremdt *et al.* (2012), we have simulated Gaussian curves $X_1(t)$ and $X_2(t)$, $t \in \mathcal{I}$, as Brownian motions (BM) or Brownian bridges (BB), and non-Gaussian (NG) curves $X_1(t)$ and $X_2(t)$, $t \in \mathcal{I}$, via

$$X_i(t) = A \sin(\pi t) + B \sin(2\pi t) + C \sin(4\pi t), \quad t \in \mathcal{I}, \quad i \in \{1, 2\}, \quad (8)$$

where $A = 7Y_1$, $B = 3Y_2$, $C = Y_3$ with Y_1 , Y_2 and Y_3 are independent t_5 -distributed random variables. All curves were simulated at 500 equidistant points in the unit interval \mathcal{I} , and transformed into functional objects using the Fourier basis with 49 basis functions. For each data generating process, we considered 500 replications. For practical and computational reasons, we have concentrated our analysis to sample sizes ranging from $n_1 = n_2 = 25$ to $n_1 = n_2 = 100$ random curves in each

group, using also the three most common nominal levels α , i.e., $\alpha \in \{0.01, 0.05, 0.10\}$. All bootstrap calculations are based on $B = 1000$ bootstrap replications.

We first illustrate the quality of the asymptotic and of the bootstrap approximations to the distribution of interest by considering the test statistic $T_{2,N}$. For this, we first estimate the exact distribution of this test statistic under the null hypothesis by generating 10,000 replications of functional data \mathbf{X}_N using (8) and $n_1 = n_2 = 25$ observations. We then compare the kernel density estimate of this exact distribution (obtained using a Gaussian kernel with bandwidth equal to 0.45) with that of the asymptotic χ_3^2 distribution and that of the bootstrap approximation using the algorithm described in Section 2.2.1 to generate the bootstrap pseudo-functional data \mathbf{X}_N^* . Figure 6.1 and 6.2 presents the results obtained by applying the bootstrap to five randomly selected samples \mathbf{X}_N .

Please insert Figure 6.1 and Figure 6.2 about here

As it seen from these exhibits, which present density estimates of the distribution of interest and corresponding QQ-plots, the density of the asymptotic χ_3^2 distribution does not provide an accurate approximation of the exact density of interest. In particular, it overestimates the exact density in the crucial region of the right tail of this distribution, as it is clearly seen in Figure 6.1. This overestimation implies that the $T_{2,N}$ test using χ^2 critical values will lead to rejection rates that are below the desired nominal size of α , that is the test will be conservative. On the other hand, and compared to the χ^2 approximation, the bootstrap estimations are much more accurate and provide a very good approximation of the exact distribution of the test statistic considered. This behavior is also clearly seen in the QQ-plots presented in Figure 6.2 .

We next investigate the sizes behavior of the tests $T_{p,N}^{(G)}$ and $T_{p,N}$ for testing equality of two covariance functions, both for BM and NG data, using the asymptotic χ^2 -approximation, where the corresponding tests are denoted by $T_{p,N}^{(G)}$ -Asym and $T_{p,N}$ -Asym, respectively and their bootstrap approximations, where the corresponding tests are denoted by $T_{p,N}^{(G)}$ -Boot and $T_{p,N}$ -Boot, respectively. We have also tested the performance of the bootstrap approximation test T_N^* , denoted by T_N -Boot. We use either $n_1 = n_2 = 25$, $n_1 = n_2 = 50$ or $n_1 = n_2 = 100$ curves, with either two ($p = 2$) or three ($p = 3$) FPC's to perform the tests.

Please insert Table 6.1 about here

Table 6.1 shows the empirical sizes obtained. As it is evident from this table, the test $T_{p,N}^{(G)}$ -Asym has a severely inflated size in the case of NG data, due to violation of the assumption of normality, a behavior which was pointed out also in the simulation study of Fremdt *et al.* (2012, Section 4). It is also evident that the test $T_{p,N}$ -Asym has a severely under-estimated size, confirming the visual evidence of Figure 6.1 and Figure 6.2. On the other hand, it is clear that the tests $T_{p,N}^{(G)}$ -Boot and

$T_{p,N}$ -Boot based on the bootstrap approximations have a very good size behavior and do not suffer from the over- or under- rejection problems from which the tests $T_{p,N}^{(G)}$ and $T_{p,N}$ based on asymptotic approximations suffer. Notice that this very good size behavior is true even for sample sizes as small as $n_1 = n_2 = 25$ observations that have been considered in the simulation study. Finally, and in contrast to the behavior of the test $T_{p,N}^{(G)}$ -Asym, notice the nice robustness property of the bootstrap-based counterpart test $T_{p,N}^{(G)}$ -Boot against deviations from Gaussianity in the case of NG data. The advantage of the bootstrap can be seen here in the overall better performance of the bootstrap-based test T_N -Boot. Recall that the test T_N does not require the choice of a truncation parameter p , referring to the number of FPC's considered, and that the asymptotic derivations of the null distribution of this test lead to results that are difficult to implement in order to calculate the critical values.

To continue our investigations of the finite sample behavior of the test statistics considered using asymptotic and bootstrap approximations, we investigate the power properties of the test $T_{p,N}$ for the case of NG data. Due to the severe size distortions of the test $T_{p,N}^{(G)}$ for NG data, we do not include this test statistic in our power study. We thus calculated the empirical rejection rates of the tests $T_{p,N}$ -Asym and $T_{p,N}$ -Boot over 500 replications, generated for either $n_1 = n_2 = 25$ or $n_1 = n_2 = 50$ observations and two ($p = 2$) FPC's. The curves in the first sample were generated according to (8) while the curves in the second sample were generated according to a scaled version of (8), i.e., $X_2(t) = \gamma X_1(t)$, $t \in \mathcal{I}$. The results are displayed for a selection of values of the scaling parameter γ , i.e., $\gamma \in \{2.0, 2.2, 2.4, 2.6, 2.8, 3.0\}$ in Table 6.2.

Please insert Table 6.2, about here

As it is evident from Table 6.2, the test $T_{p,N}$ -Boot based on bootstrap approximations has a much higher power than the test $T_{p,N}$ -Asym based on asymptotic approximations. The low power of the test $T_{p,N}$ -Asym is due to the overestimation of the right-tail of the true density, as demonstrated in Figure 6.1. Notice that, while this overestimation leads to a conservative test under the null hypothesis, it leads to a loss of power under the alternative. As can be expected, the power of the test $T_{p,N}$ -Boot improves as the deviations from the null become larger (i.e., larger values of γ 's) and/or as the sample sizes increase. Thus, and as our empirical evidence shows, the tests based on bootstrap approximations not only have a better size behavior under the null hypothesis than those based on asymptotic approximations, but they also have a much better power performance under the alternative. However, the test T_N -Boot, which does not require the choice of p (the truncation parameter), is overall the most powerful one. This clearly demonstrates the advantages of this bootstrap procedure.

We next consider the finite sample size and power properties of the asymptotic and of the bootstrap-based tests considered for testing equality of mean functions. Table 6.3 shows the

empirical sizes of the tests for the equality of two mean functions, based on the statistics $S_{p,N}^{(1)}$ -Asym and $S_{p,N}^{(2)}$ -Asym (asymptotic approximations) and $S_{p,N}^{(1)}$ -Boot, $S_{p,N}^{(2)}$ -Boot and S_N -Boot (bootstrap approximations), both for BB and NG data. Sample sizes of either $n_1 = n_2 = 25$ or $n_1 = n_2 = 50$ have been considered with either two ($p = 2$) or three ($p = 3$) FPC's and 500 replications.

Please insert Table 6.3 about here

As its evident from this table, the tests $S_{p,N}^{(1)}$ -Boot and $S_{p,N}^{(2)}$ -Boot have sizes that are quite close to the nominal ones. The same is also for the tests $S_{p,N}^{(1)}$ -Asym and $S_{p,N}^{(2)}$ -Asym, although, in most of the cases, the empirical sizes of these tests exceed the nominal ones. The advantage of the bootstrap can be seen here in the overall better performance of the bootstrap-based test S_N -Boot. Recall that the test S_N does not requires the choice of a truncation parameter p , referring to the number of FPC's considered, and that the asymptotic derivations of the null distribution of this test lead to results that are difficult to implement in order to calculate the critical values.

Finally, we investigate the power behavior of the tests considered. For this, the empirical rejection frequencies of the tests $S_{p,N}^{(1)}$ -Asym, $S_{p,N}^{(2)}$ -Asym, $S_{p,N}^{(1)}$ -Boot, $S_{p,N}^{(2)}$ -Boot and S_N -Boot have been calculated over 500 replications using NG data, either $n_1 = n_2 = 25$ or $n_1 = n_2 = 50$ curves, with two ($p = 2$) FPC's. The curves in the two samples were generated according to the model $X_i(t) = \mu_i(t) + \epsilon_i(t)$ with $\epsilon_i(t)$ generated according to (8), for $i \in \{1, 2\}$ and $t \in \mathcal{I}$. The mean functions were set equal to $\mu_1(t) = 0$ and $\mu_2(t) = \delta$ for each group respectively. The results obtained are displayed for a selection of values of the shift parameter δ , i.e., $\delta \in \{1.0, 1.2, 1.4, 1.6, 1.8, 2.0\}$ in Table 6.4.

Please insert Table 6.4 about here

As this table shows, the power results for the asymptotic-based tests $S_{p,N}^{(1)}$ -Asym and $S_{p,N}^{(2)}$ -Asym confirm the findings of Horváth & Kokoszka (2012, Table 5.1) which have been obtained for larger sample sizes and for different deviations from the null. Furthermore, the tests $S_{p,N}^{(1)}$ -Boot and $S_{p,N}^{(2)}$ -Boot show similar power behavior, although the slight better power performance of the asymptotic tests are due to the fact that these tests overestimate the nominal size, as mentioned above. The test S_N -Boot, which does not require the choice of p (the truncation parameter), is the most powerful one. This clearly demonstrates the advantages of this bootstrap procedure.

4.2 MEDITERRANEAN FRUIT FLIES

We now apply the suggested bootstrap-based testing procedure to a data set consisting of egg-laying trajectories of Mediterranean fruit flies (*Ceratitis capitata*), or medflies for short. This data set has been proved popular in the biological and statistical literature; see Müller & Stadtmüller (2005) and

references therein. It has also been analyzed by, e.g., Horváth & Kokoszka (2012, Chapter 5) (for testing the equality of two mean functions) and by Fremdt *et al.* (2013) (for testing the equality of two covariance functions).

We consider $N = 534$ egg-laying curves of medflies who lived at least 43 days, but, as in, e.g., Horváth & Kokoszka (2012, Chapter 5) and Fremdt *et al.* (2013), we only consider the egg-laying activities on the first 30 days. Two versions of these egg-laying curves are considered and are scaled such that the corresponding curves in either version are defined on the interval $\mathcal{I} = [0, 1]$. The curves in *Version 1* are denoted by $X_i(t)$ and represent the absolute counts of eggs laid by fly i on day $\lfloor 30t \rfloor$. The curves in *Version 2* are denoted by $Y_i(t)$ and represent the counts of eggs laid by fly i on day $\lfloor 30t \rfloor$ relative to the total number of eggs laid in the lifetime of fly i . Furthermore, the 534 flies are classified into short-lived flies (those who died before the end of the 43rd day after birth) and long-lived flies (those who lived 44 days or longer). In this particular data set analyzed, there are $n_1 = 256$ short-lived flies and $n_2 = 278$ long-lived flies.

Based on the above classification, we consider 2 samples. *Sample 1* represents the egg-laying curves of the *short-lived flies* $\{X_{1,i}(t) : t \in \mathcal{I}, i = 1, 2, \dots, 256\}$ (absolute curves) or $\{Y_{1,i}(t) : t \in \mathcal{I}, i = 1, 2, \dots, 256\}$ (relative curves). *Sample 2* represents the egg-laying curves of the *long-lived flies* $\{X_{2,i}(t) : t \in \mathcal{I}, i = 1, 2, \dots, 278\}$ (absolute curves) or $\{Y_{2,i}(t) : t \in \mathcal{I}, i = 1, 2, \dots, 278\}$ (relative curves). The actual curves were very irregular, hence originally smoothed slightly to produce the considered curves. Figure 6.3 shows 10 randomly selected (smoothed) curves of short-lived and long-lived flies for *Version 1* while Figure 6.4 shows 10 randomly selected (smoothed) curves of short-lived and long-lived flies for *Version 2*. The tests have been applied to such smooth curves, using a Fourier basis with 49 basis functions for the representation into functional objects. Again, all bootstrap calculations are based on $B = 1000$ bootstrap replications.

Table 6.5 shows the p -values for the absolute (Figure 6.3) and the relative (Figure 6.4) egg-laying curves of the tests for the equality of covariance functions, using the statistics $T_{1,N}$ -Asym and $T_{2,N}$ -Asym (based on asymptotic approximations) and $T_{1,N}$ -Boot, $T_{2,N}$ -Boot and T_N -Boot (based on bootstrap approximations).

Please insert Table 6.5 about here

According to these results, both tests $T_{p,N}$ -Asym and $T_{p,N}$ -Boot show in the case of the absolute egg-laying curves a uniform behavior across the range of the different values of p that explain at least 85% of the sample variance (a commonly used rule-of-thumb recommendation). In particular, and at the commonly used α -levels, the hypothesis of equality of the covariance functions cannot be rejected. However, the opposite is true for the relative egg-laying curves for which the hypothesis of equality of the covariance functions should be rejected at the commonly used α -levels and for most of the values of p considered. Notice that the bootstrap-based test $T_{p,N}$ -Boot shows in this case a more stable

behavior compared to the test $T_{p,N}$ -Asym, which does not reject the null hypothesis for large values of p . Furthermore, and due to the non-Gaussianity of the medfy data, see Fremdt *et al.* (2013, Figure 3), and the over rejection of the test $T_{p,N}^{(G)}$ -Asym demonstrated in Table 6.1, the results obtained using this test for the absolute egg-laying curves, provide little evidence that the null hypothesis of equality of covariances should be rejected. The results using the test $T_{p,N}^{(G)}$ -Boot are, however, more consistent with those obtained using the test $T_{p,N}$, designed for NG data. It is worth mentioning that the bootstrap-based test T_N -Boot, which does not require the choice of a truncation parameter p , leads to a clear rejection (for the relative egg-laying curves) and non-rejection for the absolute egg-laying curves) of the null hypothesis that the covariance functions are equal, demonstrating its usefulness in practical applications.

Please insert Table 6.6 about here

Table 6.6 shows the p -values for the absolute (Figure 6.3) egg-laying curves of the tests for the equality of mean functions. The tests used are $S_{p,N}^{(1)}$ -Asym and $S_{p,N}^{(2)}$ -Asym (based on asymptotic approximations) and $S_{p,N}^{(1)}$ -Boot, $S_{p,N}^{(2)}$ -Boot and S_N -Boot (based on bootstrap approximations). As it is evident from this table, the tests $S_{p,N}^{(2)}$ -Asym and $S_{p,N}^{(2)}$ -Boot provide a uniform behavior across the range of the first p FPC's that explain at least 85% of the sample variance, pinpointing to a rejection of the hypothesis of equality of the mean functions. The tests $S_{p,N}^{(1)}$ -Asym and $S_{p,N}^{(1)}$ -Boot show a more erratic behavior, leading to rejection of the null hypothesis in the case of small or large p and to non-rejection in the case of moderate p . This erratic behavior of the test $S_{p,N}^{(1)}$ -Asym with respect to the truncation parameter p was also pointed out in Horváth & Kokoszka (2012, Table 5.2). It is worth mentioning that the bootstrap-based test S_N -Boot, which does not require the choice of a truncation parameter p , leads to a clear rejection of the null hypothesis that the mean functions are equal, demonstrating its usefulness in practical applications. Since for the relative egg-laying curves the null hypothesis of equality of the two mean functions is rejected by all test statistics, we report only the results for the absolute egg-laying curves.

5 CONCLUSIONS

We investigated properties of a simple bootstrap-based functional testing methodology which has been applied to the important problem of comparing the mean functions and/or the covariance functions between several populations. We theoretically justified the consistency of this bootstrap testing methodology applied to some tests statistics recently proposed in the literature, and also demonstrated a very good size and power behavior in finite samples.

Although we restricted our theoretical investigations to some statistics recently proposed in the literature that build upon the empirical FPC's, the suggested bootstrap-based functional testing methodology can potentially be applied to other test statistics too. Such test statistics could be, for instance, the likelihood ratio-type statistic for testing the equality of two covariance functions considered in Gaines *et al.* (2011) or the regularized-based M -test considered in Kraus & Panaretos (2012) for the same problem. Also, subject to appropriate modifications, we conjecture that the suggested basic resampling algorithm can be adapted to different testing problems related to the comparisons of population characteristics like testing equality of distributions; see for instance the approaches of Hall & Van Keilegom (2007) and Benko *et al.* (2009). However, all the above investigations require careful attention that is beyond the scope of the present work.

6 APPENDIX: PROOFS

We first fix some notation. Let $\hat{\Delta}_N^* = \hat{A}_{1,n_1}^* - \hat{A}_{2,n_2}^* = ((\hat{A}_{1,n_1}^*(r, m) - \hat{A}_{2,n_2}^*(r, m)), 1 \leq r, m \leq p)$, where

$$\hat{A}_{1,n_1}^*(r, m) = \frac{1}{n_1} \sum_{j=1}^{n_1} \hat{a}_{1,j}^*(r) \hat{a}_{1,j}^*(m), \quad \hat{A}_{2,n_2}^*(r, m) = \frac{1}{n_2} \sum_{j=1}^{n_2} \hat{a}_{2,j}^*(r) \hat{a}_{2,j}^*(m),$$

$$\hat{a}_{1,j}^*(r) = \langle X_{1,j}^* - \bar{X}_{1,n_1}^*, \hat{\varphi}_r^* \rangle, \quad \hat{a}_{2,j}^*(r) = \langle X_{2,j}^* - \bar{X}_{2,n_2}^*, \hat{\varphi}_r^* \rangle$$

and $\bar{X}_{i,n_i}^*(t) = n_i^{-1} \sum_{j=1}^{n_i} X_{i,j}^*(t)$, $t \in \mathcal{I}$, $i = 1, 2$. Furthermore, $\hat{\lambda}_r^*$ and $\hat{\varphi}_r^*$, $r = 1, 2, \dots, N$, denote the eigenvalues and eigenfunctions, respectively, of the pooled covariance matrix $\hat{C}_N^*(t, s) = (n_1/N) \hat{C}_{1,n_1}^*(t, s) + (n_2/N) \hat{C}_{2,n_2}^*(t, s)$, where $\hat{C}_{i,n_i}^* = n_i^{-1} \sum_{j=1}^{n_i} (X_{i,j}^*(t) - \bar{X}_{i,n_i}^*(t))(X_{i,j}^*(s) - \bar{X}_{i,n_i}^*(s))$, $t, s \in \mathcal{I}$, $i = 1, 2$. We assume that $\hat{\lambda}_1^* \geq \hat{\lambda}_2^* \geq \hat{\lambda}_3^* \geq \dots \geq \hat{\lambda}_N^*$, and recall that $N = n_1 + n_2$. We first establish the following useful lemmas.

Lemma 6.1 *Under the assumptions of Theorem 3.3 we have, conditionally on \mathbf{X}_N , that, for $1 \leq i \leq p$,*

$$(i) \quad |\hat{\lambda}_i^* - \hat{\lambda}_i| = O_P(N^{-1/2}) \quad \text{and} \quad (ii) \quad \|\hat{\varphi}_i^* - \hat{c}_i^* \hat{\varphi}_i\| = O_P(N^{-1/2}),$$

where $\hat{c}_i^* = \text{sign}(\langle \hat{\varphi}_i^*, \hat{\varphi}_i \rangle)$.

Proof: We first show that for $r \in \{1, 2\}$,

$$\left\| n_r^{-1/2} \sum_{j=1}^{n_r} \{ (X_{r,j}^*(t) - \bar{X}_{r,n_r}^*(t))(X_{r,j}^*(s) - \bar{X}_{r,n_r}^*(s)) - \hat{C}_N(t, s) \} \right\| = O_P(1) \quad (9)$$

and

$$\left\| n_r^{-1/2} \sum_{j=1}^{n_r} \{ X_{r,j}^*(t) - \bar{X}_{r,n_r}^*(t) \} \right\| = O_P(1). \quad (10)$$

Let $Y_{r,j}^*(t) = X_{r,j}^*(t) - \bar{X}_{r,n_r}^*(t)$ and $\tilde{Y}_{r,j}^*(t) = X_{r,j}^*(t) - \bar{X}_{r,n_r}(t)$. Then

$$\begin{aligned} \mathbb{E} \left\| n_r^{-1/2} \sum_{j=1}^{n_r} \{Y_{r,j}^*(t)Y_{r,j}^*(s) - \hat{C}_N(t,s)\} \right\|^2 &\leq \frac{2}{n_r} \int_0^1 \int_0^1 \sum_{j=1}^{n_r} \mathbb{E} \left(\tilde{Y}_{r,j}^*(t)\tilde{Y}_{r,j}^*(s) - \hat{C}_N(t,s) \right)^2 dt ds \\ &\quad + 2 \mathbb{E} \left\| n_r^{-1/2} \sum_{j=1}^{n_r} \{Y_{r,j}^*(t)Y_{r,j}^*(s) - \tilde{Y}_{r,j}^*(t)\tilde{Y}_{r,j}^*(s)\} \right\|^2 \\ &= 2 \int_0^1 \int_0^1 \mathbb{E} \left(\varepsilon_{r,1}^*(t)\varepsilon_{r,1}^*(s) - \hat{C}_N(t,s) \right)^2 dt ds + O_P(1), \end{aligned}$$

using the fact that $Y_{r,j}^*(t) = \tilde{Y}_{r,j}^*(t) + (\bar{X}_{r,n_r}(t) - \bar{X}_{r,n_r}^*(t)) = \varepsilon_{r,j}^*(t) + O_P(n_r^{-1/2})$. Now, since $\mathbb{E}(\varepsilon_{r,1}^*(t)\varepsilon_{r,1}^*(s) - \hat{C}_N(t,s))^2 = N^{-1} \sum_{r=1}^2 \sum_{j=1}^{n_r} [\hat{\varepsilon}_{r,j}(t)\hat{\varepsilon}_{r,j}(s) - \hat{C}_N(t,s)]^2 = O_P(1)$, assertion (9) follows by Markov's inequality. Assertion (10) follows by the same inequality and because

$$\begin{aligned} \mathbb{E} \left\| n_r^{-1/2} \sum_{j=1}^{n_r} \{X_{r,j}^*(t) - \bar{X}_{r,n_r}^*(t)\} \right\|^2 &\leq 2 \mathbb{E} \left\| n_r^{-1/2} \sum_{j=1}^{n_r} \tilde{Y}_{r,j}^* \right\|^2 + O_P(1) \\ &= \int_0^1 \mathbb{E}(\varepsilon_{r,1}^*(t))^2 dt + O_P(1) = O_P(1). \end{aligned}$$

Using (9), we get that

$$\|\hat{C}_N^* - \hat{C}_N\|_S \leq \frac{n_1}{N} \|\hat{C}_{1,n_1}^* - \hat{C}_N\|_S + \frac{n_2}{N} \|\hat{C}_{2,n_2}^* - \hat{C}_N\|_S = \frac{n_1}{N} O_P(n_1^{-1/2}) + \frac{n_2}{N} O_P(n_2^{-1/2}) = O_P(N^{-1/2}). \quad (11)$$

Using (11) and Lemmas 2.2 and 2.3 of Horváth & Kokoszka (2012), we have that, for $1 \leq i \leq p$,

$$|\hat{\lambda}_i^* - \hat{\lambda}_i| \leq \|\hat{C}_N^* - \hat{C}_N\|_S = O_P(N^{-1/2}),$$

and

$$\|\hat{\varphi}_i^* - \hat{c}_i^* \hat{\varphi}_i\| = O_P(\|\hat{C}_N^* - \hat{C}_N\|_S) = O_P(N^{-1/2}).$$

This completes the proof of the lemma. \square

Let

$$\begin{aligned} \tilde{A}_{1,n_1}^*(i,j) &= \frac{1}{n_r} \sum_{k=1}^{n_r} \langle X_{1,k}^* - \bar{X}_{1,n_1}, \hat{c}_i^* \hat{\varphi}_i \rangle \langle X_{1,k}^* - \bar{X}_{1,n_1}, \hat{c}_j^* \hat{\varphi}_j \rangle, \\ \tilde{A}_{2,n_2}^*(i,j) &= \frac{1}{n_2} \sum_{k=1}^{n_2} \langle X_{2,k}^* - \bar{X}_{2,n_2}, \hat{c}_i^* \hat{\varphi}_i \rangle \langle X_{2,k}^* - \bar{X}_{2,n_2}, \hat{c}_j^* \hat{\varphi}_j \rangle \end{aligned}$$

and $\hat{\Delta}_N^* = ((\hat{\Delta}_N^*(i,j)), 1 \leq i,j \leq p)$, where $\hat{\Delta}_N^*(i,j) = (\tilde{A}_{1,n_1}^*(i,j) - \tilde{A}_{2,n_2}^*(i,j))$.

Lemma 6.2 *Under the assumptions of Theorem 3.3 we have, conditionally on \mathbf{X}_N , that*

$$\sqrt{\frac{n_1 n_2}{N}} (\hat{\Delta}_N^* - \tilde{\Delta}_N^*) = o_P(1).$$

Proof: Let $\check{a}_{r,j}^*(i) = \langle X_{r,j}^* - \bar{X}_{r,n_r}, \hat{\varphi}_i^* \rangle$, $\tilde{a}_{r,j}^*(i) = \langle X_{r,j}^* - \bar{X}_{r,n_r}, \hat{c}_i^* \hat{\varphi}_i \rangle$. We first show that we can replace $\hat{a}_{r,j}^*(i)$ in $\hat{\Delta}_N^*(i, j)$ by $\check{a}_{r,j}^*(i)$. For this, notice that, for $r \in \{1, 2\}$, we have

$$\begin{aligned} \sqrt{\frac{n_1 n_2}{N}} \frac{1}{n_r} \sum_{k=1}^{n_r} (\hat{a}_{r,k}^*(i) \hat{a}_{r,k}^*(j) - \check{a}_{r,k}^*(i) \check{a}_{r,k}^*(j)) &= \sqrt{\frac{n_1 n_2}{N}} \frac{1}{n_r} \sum_{k=1}^{n_r} (\hat{a}_{r,k}^*(i) - \check{a}_{r,k}^*(i)) \hat{a}_{r,k}^*(j) \\ &\quad - \sqrt{\frac{n_1 n_2}{N}} \frac{1}{n_r} \sum_{k=1}^{n_r} (\check{a}_{r,k}^*(j) - \hat{a}_{r,k}^*(j)) \check{a}_{r,k}^*(i) \end{aligned}$$

and

$$\begin{aligned} \sqrt{\frac{n_1 n_2}{N}} \frac{1}{n_r} \sum_{k=1}^{n_r} (\hat{a}_{r,k}^*(i) - \check{a}_{r,k}^*(i)) \hat{a}_{r,k}^*(j) &= \sqrt{\frac{n_1 n_2}{N}} \int_0^1 \int_0^1 (\bar{X}_{r,n_r}(t) - \bar{X}_{r,n_r}^*(t)) \hat{\varphi}_i^*(t) dt \\ &\quad \times \frac{1}{n_r} \sum_{k=1}^{n_r} (X_{r,k}^*(s) - \bar{X}_{r,n_r}^*(s)) \hat{\varphi}_j^*(s) ds \\ &= 0. \end{aligned}$$

Let $\check{\Delta}_N^*(i, j)$ be the same expression as $\hat{\Delta}_N^*(i, j)$ with $\hat{a}_{r,j}^*(l)$ replaced by $\check{a}_{r,j}^*(l)$, $l \in \{i, j\}$, and notice that by the previous considerations, $\sqrt{n_1 n_2 / N} |\hat{\Delta}_N^*(i, j) - \check{\Delta}_N^*(i, j)| = o_P(1)$. Furthermore,

$$\begin{aligned} \sqrt{\frac{n_1 n_2}{N}} (\check{\Delta}_N^*(i, j) - \tilde{\Delta}_N^*(i, j)) &= \sqrt{\frac{n_1 n_2}{N}} \int_0^1 \int_0^1 \frac{1}{n_1} \sum_{k=1}^{n_1} [(X_{1,k}^*(t) - \bar{X}_{1,n_1}(t))(X_{1,k}^*(s) - \bar{X}_{1,n_1}(s)) - \hat{C}_N(t, s)] \\ &\quad \times (\hat{\varphi}_i^*(t) \hat{\varphi}_j^*(s) - \hat{c}_i^* \hat{c}_j^* \hat{\varphi}_i(t) \hat{\varphi}_j(s)) dt ds \\ &\quad - \sqrt{\frac{n_1 n_2}{N}} \int_0^1 \int_0^1 \frac{1}{n_2} \sum_{k=1}^{n_2} [(X_{2,k}^*(t) - \bar{X}_{2,n_2}(t))(X_{2,k}^*(s) - \bar{X}_{2,n_2}(s)) - \hat{C}_N(t, s)] \\ &\quad \times (\hat{\varphi}_i^*(t) \hat{\varphi}_j^*(s) - \hat{c}_i^* \hat{c}_j^* \hat{\varphi}_i(t) \hat{\varphi}_j(s)) dt ds \\ &= V_{1,N} + V_{2,N}, \end{aligned}$$

with an obvious notation for $V_{r,N}$, $r \in \{1, 2\}$. Now,

$$\begin{aligned} |V_{r,N}| &\leq \sqrt{\frac{n_1 n_2}{N}} \int_0^1 \int_0^1 \left| \frac{1}{n_r} \sum_{k=1}^{n_r} [(X_{r,k}^*(t) - \bar{X}_{r,n_r}(t))(X_{r,k}^*(s) - \bar{X}_{r,n_r}(s)) - \hat{C}_N(t, s)] \right. \\ &\quad \times (\hat{\varphi}_i^*(t) - \hat{c}_i^* \hat{\varphi}_i(t)) \hat{\varphi}_j^*(s) \Big| dt ds \\ &\quad + \sqrt{\frac{n_1 n_2}{N}} \int_0^1 \int_0^1 \left| \frac{1}{n_r} \sum_{k=1}^{n_r} [(X_{r,k}^*(t) - \bar{X}_{r,n_r}(t))(X_{r,k}^*(s) - \bar{X}_{r,n_r}(s)) - \hat{C}_N(t, s)] \right. \\ &\quad \times (\hat{c}_j^* \hat{\varphi}_j(s) - \hat{\varphi}_j^*(t)) \hat{c}_i^* \hat{\varphi}_i(t) \Big| dt ds \\ &\leq \frac{1}{\sqrt{n_r}} \sqrt{\frac{n_1 n_2}{N}} \left\| \frac{1}{\sqrt{n_r}} \sum_{k=1}^{n_r} [(X_{r,k}^*(t) - \bar{X}_{r,n_r}(t))(X_{r,k}^*(s) - \bar{X}_{r,n_r}(s)) - \hat{C}_N(t, s)] \right\| \\ &\quad \times \left\{ \|\hat{\varphi}_i^* - \hat{c}_i^* \hat{\varphi}_i\| + \|\hat{\varphi}_j^* - \hat{c}_j^* \hat{\varphi}_j\| \right\} \\ &= O_P\left(\frac{1}{\sqrt{n_r}} \sqrt{\frac{n_1 n_2}{N}}\right) \left\{ O_P(\|\hat{\varphi}_i^* - \hat{c}_i^* \hat{\varphi}_i\|) + O_P(\|\hat{\varphi}_j^* - \hat{c}_j^* \hat{\varphi}_j\|) \right\} = O_P(N^{-1/2}), \end{aligned}$$

because of (9) and Lemma 6.1. This completes the proof of the lemma. \square

Proof of Theorem 3.1: Let \mathcal{S} be the Hilbert space of Hilbert–Schmidt operators endowed with the inner product $\langle \Psi_1, \Psi_2 \rangle_{\mathcal{S}} = \sum_{j=1}^{\infty} \langle \Psi_1(e_j), \Psi_2(e_j) \rangle$ for $\Psi_1, \Psi_2 \in \mathcal{S}$, where $\{e_j : j = 1, 2, \dots\}$ is an orthonormal basis in \mathcal{H} . Notice that $\widehat{\mathcal{C}}_i^* \in \mathcal{S}$, $i = 1, 2$. Since

$$\overline{X}_{i,n_i}^* = \overline{X}_{i,n_i} + O_P(n_i^{-1/2}) \quad \text{and} \quad n_i^{-1/2} \sum_{j=1}^{n_i} (X_{i,j}^* - \overline{X}_{i,n_i}^*) = O_P(1), \quad i = 1, 2,$$

we get

$$\begin{aligned} \widehat{\mathcal{C}}_i^* &= \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{i,j}^* - \overline{X}_{i,n_i}^*) \otimes (X_{i,j}^* - \overline{X}_{i,n_i}^*) \\ &= \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{i,j}^* - \overline{X}_{i,n_i}) \otimes (X_{i,j}^* - \overline{X}_{i,n_i}) + O_P(n^{-1}), \quad i = 1, 2, \end{aligned}$$

where the random variables $(X_{i,j}^* - \overline{X}_{i,n_i}) \otimes (X_{i,j}^* - \overline{X}_{i,n_i})$ are, conditional on X_N , independent and identically distributed. By a central limit theorem for triangular arrays of independent and identically distributed \mathcal{S} -valued random variables (see, e.g., Politis & Romano (1992, Theorem 4.2)), we get, conditionally on X_N , that $\sqrt{n_i}(\widehat{\mathcal{C}}_i^* - \widehat{\mathcal{C}}_N)$ converges weakly to a Gaussian random element \mathcal{U} in \mathcal{S} with mean zero and covariance operator $\mathcal{B} = \theta_1 \mathcal{B}_1 + (1 - \theta_1) \mathcal{B}_2$ as $n_i \rightarrow \infty$. Here, \mathcal{B}_i is the covariance operator of the limiting Gaussian random element U_i to which $\sqrt{n_i}(\widehat{\mathcal{C}}_i - \mathcal{C}_i)$ converges weakly as $n_i \rightarrow \infty$.

By the independence of the bootstrap random samples between the two populations, we have, conditional on X_N ,

$$\begin{aligned} T_N^* &= N \|\widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_2^*\|_{\mathcal{S}}^2 \\ &= N \langle \widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_N, \widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_N \rangle_{\mathcal{S}} + N \langle \widehat{\mathcal{C}}_2^* - \widehat{\mathcal{C}}_N, \widehat{\mathcal{C}}_2^* - \widehat{\mathcal{C}}_N \rangle_{\mathcal{S}} \\ &= \frac{N}{n_1} \|\sqrt{n_1}(\widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_N)\|_{\mathcal{S}}^2 + \frac{N}{n_2} \|\sqrt{n_2}(\widehat{\mathcal{C}}_2^* - \widehat{\mathcal{C}}_N)\|_{\mathcal{S}}^2. \end{aligned}$$

Hence, taking into account the above results and that $n_1/N \rightarrow \theta_1$, we have that $N\|\widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_2^*\|_{\mathcal{S}}^2$ converges weakly to $\sum_{l=1}^{\infty} \tilde{\lambda}_l Z_l^2$ as $n_1, n_2 \rightarrow \infty$, where $\tilde{\lambda}_l$, $l \geq 1$, are the eigenvalues of the operator $\tilde{\mathcal{B}} = \theta_1^{-1} \mathcal{B} + (1 - \theta_1)^{-1} \mathcal{B}$ and Z_l , $l \geq 1$, are independent standard (real-valued) Gaussian distributed random variables. Since $\mathcal{B}_1 = \mathcal{B}_2$, the assertion follows. \square

Proof of Theorem 3.3: Recall that $T_{p,N}^* = (n_1 n_2 / N) \widehat{\xi}_N^{*'} \widehat{L}_N^{*-1} \widehat{\xi}_N^*$ with $\widehat{\xi}_N^* = \text{vech}(\widehat{\Delta}_N^*)$ and \widehat{L}_N^* is an estimator of the covariance matrix of $\sqrt{n_1 n_2 / N} \widehat{\xi}_N^*$. The element of the latter matrix corresponding to the covariance of $\sqrt{n_1 n_2 / N} \widehat{\xi}_N^*(i_1, j_1)$ and $\sqrt{n_1 n_2 / N} \widehat{\xi}_N^*(i_2, j_2)$, $1 \leq i_1 \leq j_1 \leq p$ and $1 \leq i_2 \leq j_2 \leq p$,

is denoted by $l(\widehat{\xi}_N^*(i_1, j_1), \widehat{\xi}_N^*(i_2, j_2))$ and is estimated by

$$\begin{aligned} \widehat{l}(\widehat{\xi}_N^*(i_1, j_1), \widehat{\xi}_N^*(i_2, j_2)) &= \frac{n_2}{n_1 + n_2} \left\{ \frac{1}{n_1} \sum_{j=1}^{n_1} \widehat{a}_{1,j}^*(i_1) \widehat{a}_{1,j}^*(j_1) \widehat{a}_{1,j}^*(i_2) \widehat{a}_{1,j}^*(j_2) - \langle \widehat{C}_{1,n_1}^* \widehat{\varphi}_{i_1}^*, \widehat{\varphi}_{j_1}^* \rangle \right. \\ &\quad \times \langle \widehat{C}_{1,n_1}^* \widehat{\varphi}_{i_2}^*, \widehat{\varphi}_{j_2}^* \rangle \left. + \frac{n_1}{n_1 + n_2} \left\{ \frac{1}{n_2} \sum_{j=1}^{n_2} \widehat{a}_{2,j}^*(i_1) \widehat{a}_{2,j}^*(j_1) \widehat{a}_{2,j}^*(i_2) \widehat{a}_{2,j}^*(j_2) \right. \right. \\ &\quad \left. \left. - \langle \widehat{C}_{2,n_2}^* \widehat{\varphi}_{i_1}^*, \widehat{\varphi}_{j_1}^* \rangle \langle \widehat{C}_{2,n_2}^* \widehat{\varphi}_{i_2}^*, \widehat{\varphi}_{j_2}^* \rangle \right\} \right\}. \end{aligned}$$

Let $\widehat{\Delta}_N(r, m) = \widehat{A}_{1,n_1}(r, m) - \widehat{A}_{2,n_2}(r, m)$ be the (r, m) th element of the matrix $\widehat{\Delta}_N^*$. To establish the theorem, it suffices to show that, under the assumptions made, the following assertions (12) and (13) are true.

$$\mathcal{L}\left(\left(\sqrt{\frac{n_1 n_2}{N}} \widehat{\Delta}_N^*(i, j), 1 \leq i, j \leq p\right) \middle| \mathbf{X}_N\right) \Rightarrow \mathcal{L}((\Delta(i, j), 1 \leq i, j \leq p)), \quad (12)$$

where $\Delta = (\Delta(i, j), 1 \leq i, j \leq p)$ is a Gaussian random matrix with $\mathbb{E}(\Delta(i, j)) = 0$, $1 \leq i, j \leq p$ having a positive definite covariance matrix Σ with elements $\sigma(i_1, j_1, i_2, j_2) = \text{Cov}(\Delta(i_1, j_1), \Delta(i_2, j_2))$, $1 \leq i_1, i_2, j_1, j_2 \leq p$, given by

$$\begin{aligned} \sigma(i_1, j_1, i_2, j_2) &= (1 - \theta) \left\{ \mathbb{E}(\langle X_{1,j} - \mu_1, \varphi_{i_1} \rangle \langle X_{1,j} - \mu_1, \varphi_{j_1} \rangle \langle X_{1,j} - \mu_1, \varphi_{i_2} \rangle \langle X_{1,j} - \mu_1, \varphi_{j_2} \rangle) \right. \\ &\quad \left. - \langle C \varphi_{i_1}, \varphi_{j_1} \rangle \langle C \varphi_{i_2}, \varphi_{j_2} \rangle \right\} \\ &\quad + \theta \left\{ \mathbb{E}(\langle X_{2,j} - \mu_2, \varphi_{i_1} \rangle \langle X_{2,j} - \mu_2, \varphi_{j_1} \rangle \langle X_{2,j} - \mu_2, \varphi_{i_2} \rangle \langle X_{2,j} - \mu_2, \varphi_{j_2} \rangle) \right. \\ &\quad \left. - \langle C \varphi_{i_1}, \varphi_{j_1} \rangle \langle C \varphi_{i_2}, \varphi_{j_2} \rangle \right\}, \end{aligned}$$

where $C = (1 - \theta)C_1 + \theta C_2$. Furthermore,

$$\widehat{l}(\widehat{\xi}_N^*(i_1, j_1), \widehat{\xi}_N^*(i_2, j_2)) - \widehat{c}_{i_1}^* \widehat{c}_{j_1}^* \widehat{c}_{i_2}^* \widehat{c}_{j_2}^* \sigma(i_1, j_1, i_2, j_2) \rightarrow 0, \quad \text{in probability,} \quad (13)$$

for all $1 \leq i_1, i_2, j_1, j_2 \leq p$.

To establish (12), recall that by Lemma 6.2 it suffices to consider the asymptotic distribution of $\sqrt{n_1 n_2 / N} \widehat{\Delta}_N^*$. Let $\widetilde{Y}_{r,k}^*(i, j) = \langle X_{r,k}^* - \overline{X}_{r,n_r}, \widehat{c}_i^* \widehat{\varphi}_i \rangle \langle X_{r,k}^* - \overline{X}_{r,n_r}, \widehat{c}_j^* \widehat{\varphi}_j \rangle$ and notice that

$$\sqrt{n_1 n_2 / N} \widehat{\Delta}_N^*(i, j) = \sqrt{n_1 n_2 / N} \left(\frac{1}{n_1} \sum_{k=1}^{n_1} \widetilde{Y}_{1,k}^*(i, j) - \frac{1}{n_2} \sum_{k=1}^{n_2} \widetilde{Y}_{2,k}^*(i, j) \right).$$

Since \widehat{c}_i^* and \widehat{c}_j^* change solely the sign of $\widetilde{Y}_{r,k}^*(i, j)$, they do not affect the limiting distribution of the two sums above. Thus, without loss of generality, we set $\widehat{c}_i^* = \widehat{c}_j^* = 1$. Let $\check{Y}_{r,k}^*(i, j) = \langle X_{r,k}^* - \overline{X}_{r,n_r}, \widehat{\varphi}_i \rangle \langle X_{r,k}^* - \overline{X}_{r,n_r}, \widehat{\varphi}_j \rangle$, notice that $\mathbb{E}(\check{Y}_{r,k}^*(i, j)) = \widehat{\lambda}_i \mathbf{1}_{\{i=j\}}$, and consider instead of the distribution of

$\sqrt{n_1 n_2 / N} \tilde{\Delta}_N^*$ the distribution of the asymptotically equivalent sum $\sqrt{n_1 n_2 / N} Z_N^*(i, j)$, given by

$$\begin{aligned} \sqrt{n_1 n_2 / N} Z_N^*(i, j) &= \sqrt{n_1 n_2 / N} \left(\frac{1}{n_1} \left(\sum_{k=1}^{n_1} \check{Y}_{1,k}^*(i, j) \right) - \frac{1}{n_2} \sum_{k=1}^{n_2} \check{Y}_{2,k}^*(i, j) \right) \\ &= \sqrt{n_2 / N} \frac{1}{\sqrt{n_1}} \sum_{k=1}^{n_1} (\check{Y}_{1,k}^*(i, j) - \mathbf{1}_{\{i=j\}} \hat{\lambda}_i) - \sqrt{n_1 / N} \frac{1}{\sqrt{n_2}} \sum_{k=1}^{n_2} (\check{Y}_{2,k}^*(i, j) - \mathbf{1}_{\{i=j\}} \hat{\lambda}_i) \\ &= \sqrt{n_2 / N} Z_{1,N}^*(i, j) - \sqrt{n_1 / N} Z_{2,N}^*(i, j), \end{aligned}$$

with an obvious notation for $Z_{r,N}^*(i, j)$, $r \in \{1, 2\}$. Notice that, conditionally on \mathbf{X}_N , $Z_N^*(i, j)$ is distributed as the difference of the two independent sums $Z_{1,N}^*(i, j)$ and $Z_{2,N}^*(i, j)$, where for $r \in \{1, 2\}$, $Z_{r,N}^*(i, j)$ is a sum of the independent and identically distributed random variables $\check{Y}_{r,k}^*(i, j) - \hat{\lambda}_i \mathbf{1}_{\{i=j\}}$, $k = 1, 2, \dots, n_r$. Furthermore, $\mathbb{E}(Z_{r,N}^*(i, j)) = 0$ and since $\varepsilon_{r,k}^* = X_{r,k}^* - \bar{X}_{r,n_r}$, we get

$$\begin{aligned} \text{Cov}(Z_{r,k}(i_1, j_1), Z_{r,k}(i_2, j_2)) &= \mathbb{E}(Z_{r,k}(i_1, j_1) Z_{r,k}(i_2, j_2)) \\ &= \mathbb{E} \left[\langle \varepsilon_{1,k}^*, \hat{\varphi}_{i_1} \rangle \langle \varepsilon_{1,k}^*, \hat{\varphi}_{j_1} \rangle \langle \varepsilon_{1,k}^*, \hat{\varphi}_{i_2} \rangle \langle \varepsilon_{1,k}^*, \hat{\varphi}_{j_2} \rangle \right] \\ &\quad - \hat{\lambda}_{i_1} \hat{\lambda}_{i_2} \mathbf{1}_{\{i_1=j_1\}} \mathbf{1}_{\{i_2=j_2\}} \\ &= \sum_{r=1}^2 \frac{n_r}{N} \frac{1}{n_r} \sum_{k=1}^{n_r} \langle \hat{\varepsilon}_{r,k}, \hat{\varphi}_{i_1} \rangle \langle \hat{\varepsilon}_{r,k}, \hat{\varphi}_{j_1} \rangle \langle \hat{\varepsilon}_{r,k}, \hat{\varphi}_{i_2} \rangle \langle \hat{\varepsilon}_{r,k}, \hat{\varphi}_{j_2} \rangle \\ &\quad - \hat{\lambda}_{i_1} \hat{\lambda}_{i_2} \mathbf{1}_{\{i_1=j_1\}} \mathbf{1}_{\{i_2=j_2\}} \\ &\rightarrow (1 - \theta) \mathbb{E} \left[\langle X_{1,k} - \mu_1, \varphi_{i_1} \rangle \langle X_{1,k} - \mu_1, \varphi_{j_1} \rangle \langle X_{1,k} - \mu_1, \varphi_{i_2} \rangle \right. \\ &\quad \times \left. \langle X_{1,k} - \mu_1, \varphi_{j_2} \rangle \right] + \theta \mathbb{E} \left[\langle X_{2,k} - \mu_2, \varphi_{i_1} \rangle \langle X_{2,k} - \mu_2, \varphi_{j_1} \rangle \right. \\ &\quad \times \left. \langle X_{2,k} - \mu_2, \varphi_{i_2} \rangle \langle X_{2,k} - \mu_2, \varphi_{j_2} \rangle \right] - \lambda_{i_1} \lambda_{i_2} \mathbf{1}_{\{i_1=j_1\}} \mathbf{1}_{\{i_2=j_2\}} \\ &= \sigma(i_1, j_1, i_2, j_2), \end{aligned} \tag{14}$$

in probability, by the weak law of large numbers and using $\langle \hat{\varepsilon}_{r,l}, \hat{\varphi}_i \rangle = \langle X_{r,l} - \bar{X}_{r,n_r}, \hat{\varphi}_i \rangle$. Thus, by a multivariate central limit theorem for triangular arrays of real valued random vectors, we get that $\mathcal{L}(Z_{r,N}^*) \Rightarrow Z$, where Z is a Gaussian distributed $p \times p$ random matrix, with $\mathbb{E}(Z(i, j)) = 0$ and $\text{Cov}(Z(i_1, j_1), Z(i_2, j_2)) = \sigma(i_1, j_1, i_2, j_2)$. To conclude the proof of (12), notice that

$$\mathcal{L}(Z_N^*) = \mathcal{L}(\sqrt{n_2 / N} Z_{1,N}^* + \sqrt{n_1 / N} Z_{2,N}^*) \Rightarrow \mathcal{L}(\sqrt{1 - \theta} Z_1 + \sqrt{\theta} Z_2) = \mathcal{L}(Z),$$

where Z_1 and Z_2 are two independent copies of the Gaussian random matrix Z .

To establish (13), notice that, for $r \in \{1, 2\}$,

$$\frac{1}{n_r} \sum_{j=1}^{n_r} \hat{a}_{r,j}^*(i_1) \hat{a}_{r,j}^*(j_1) \hat{a}_{r,j}^*(i_2) \hat{a}_{r,j}^*(j_2) = \frac{1}{n_r} \sum_{j=1}^{n_r} \tilde{a}_{r,j}^*(i_1) \tilde{a}_{r,j}^*(j_1) \tilde{a}_{r,j}^*(i_2) \tilde{a}_{r,j}^*(j_2) + O_P(n_r^{-1/2})$$

and that, for

$$\begin{aligned} \sigma^{(1)}(i_1, j_1, i_2, j_2) = & (1 - \theta) \mathbb{E}[\langle X_{1,k} - \mu_1, \varphi_{i_1} \rangle \langle X_{1,k} - \mu_1, \varphi_{j_1} \rangle \langle X_{1,k} - \mu_1, \varphi_{i_2} \rangle \langle X_{1,k} - \mu_1, \varphi_{j_2} \rangle] \\ & + \theta \mathbb{E}[\langle X_{2,k} - \mu_2, \varphi_{i_1} \rangle \langle X_{2,k} - \mu_2, \varphi_{j_1} \rangle \langle X_{2,k} - \mu_2, \varphi_{i_2} \rangle \langle X_{2,k} - \mu_2, \varphi_{j_2} \rangle], \end{aligned}$$

we have

$$\frac{1}{n_r} \sum_{j=1}^{n_r} \tilde{a}_{r,j}^*(i_1) \tilde{a}_{r,j}^*(j_1) \tilde{a}_{r,j}^*(i_2) \tilde{a}_{r,j}^*(j_2) - \tilde{c}_{i_1}^* \tilde{c}_{j_1}^* \tilde{c}_{i_2}^* \tilde{c}_{j_2}^* \sigma^{(1)}(i_1, j_1, i_2, j_2) \rightarrow 0, \quad (15)$$

in probability, since as in obtaining (14),

$$\begin{aligned} & \mathbb{E} \left(\frac{1}{n_r} \sum_{j=1}^{n_r} \tilde{a}_{r,j}^*(i_1) \tilde{a}_{r,j}^*(j_1) \tilde{a}_{r,j}^*(i_2) \tilde{a}_{r,j}^*(j_2) \right) - \tilde{c}_{i_1}^* \tilde{c}_{j_1}^* \tilde{c}_{i_2}^* \tilde{c}_{j_2}^* \sigma^{(1)}(i_1, j_1, i_2, j_2) \\ & = \mathbb{E} \left[\langle \varepsilon_{r,k}^*, \tilde{c}_{i_1}^* \hat{\varphi}_{i_1} \rangle \langle \varepsilon_{r,k}^*, \tilde{c}_{j_1}^* \hat{\varphi}_{j_1} \rangle \langle \varepsilon_{r,k}^*, \tilde{c}_{i_2}^* \hat{\varphi}_{i_2} \rangle \langle \varepsilon_{r,k}^*, \tilde{c}_{j_2}^* \hat{\varphi}_{j_2} \rangle \right] \\ & \quad - \tilde{c}_{i_1}^* \tilde{c}_{j_1}^* \tilde{c}_{i_2}^* \tilde{c}_{j_2}^* \sigma^{(1)}(i_1, j_1, i_2, j_2) \\ & \rightarrow 0, \end{aligned}$$

in probability, and also $Var(n_r^{-1} \sum_{j=1}^{n_r} \tilde{a}_{r,j}^*(i_1) \tilde{a}_{r,j}^*(j_1) \tilde{a}_{r,j}^*(i_2) \tilde{a}_{r,j}^*(j_2)) = O_P(n_r^{-1})$, due to the independence of the random variables $\tilde{a}_{r,j}^*(i_1) \tilde{a}_{r,j}^*(j_1) \tilde{a}_{r,j}^*(i_2) \tilde{a}_{r,j}^*(j_2)$, for different j 's. Furthermore, by the triangular inequality and because $|\langle \hat{C}_N \hat{\varphi}_i, \hat{\varphi}_j \rangle - \langle C \varphi_i, \varphi_j \rangle| \rightarrow 0$ in probability, it yields that

$$|\langle \hat{C}_{r,n_r}^* \hat{\varphi}_i^*, \hat{\varphi}_j^* \rangle - \tilde{c}_i^* \tilde{c}_j^* \langle C \varphi_i, \varphi_j \rangle| \rightarrow 0, \quad (16)$$

in probability, since

$$\begin{aligned} |\langle \hat{C}_{r,n_r}^* \hat{\varphi}_i^*, \hat{\varphi}_j^* \rangle - \tilde{c}_i^* \tilde{c}_j^* \langle \hat{C}_N \hat{\varphi}_i, \hat{\varphi}_j \rangle| & \leq |\langle \hat{C}_{r,n_r}^* - \hat{C}_N, \hat{\varphi}_i^*, \hat{\varphi}_j^* \rangle| + |\langle (\hat{C}_N (\hat{\varphi}_i^* - \tilde{c}_i^* \hat{\varphi}_i), \hat{\varphi}_j^* \rangle| \\ & \quad + |\langle (\hat{C}_N \hat{\varphi}_i, \hat{\varphi}_j^* - \tilde{c}_j^* \hat{\varphi}_j) \rangle| \\ & = O_P(\|\hat{C}_{r,n_r}^* - \hat{C}_N\| + \|\hat{\varphi}_i^* - \tilde{c}_i^* \hat{\varphi}_i\| + \|\hat{\varphi}_j^* - \tilde{c}_j^* \hat{\varphi}_j\|) \rightarrow 0, \end{aligned}$$

by (9) and Lemma 6.1. Equations (15) and (16) imply then assertion (13). This completes the proof of the theorem. \square

Proof of Theorem 3.2: Notice that under Gaussianity of the random functions $X_{1,j}$ and $X_{2,j}$, the random variables $\langle X_{1,j} - \mu_1, \varphi_i \rangle$ and $\langle X_{2,j} - \mu_2, \varphi_i \rangle$ are independent Gaussian distributed with mean zero and variance λ_i , $i = 1, 2, \dots, p$. From assertion (12) in the proof of Theorem 3.3, we get that in this case, the random variables $\Delta(i, j)$ are for $1 \leq i \leq j \leq p$ independent with mean zero and

$Var(\Delta(i, j)) = 2\lambda_i^2$ if $i = j$ and $Var(\Delta(i, j)) = \lambda_i\lambda_j$ if $i \neq j$. We then have that

$$\begin{aligned} T_{p,N}^{*(G)} &= \frac{n_1 n_2}{N} \frac{1}{2} \left(\sum_{r=1}^p \frac{\widehat{\Delta}^{*2}(r, r)}{\widehat{\lambda}_r^{*2}} + 2 \sum_{1 \leq r < m \leq p} \frac{\widehat{\Delta}^{*2}(r, m)}{\widehat{\lambda}_r^* \widehat{\lambda}_m^*} \right) \\ &= \sum_{r=1}^p \left(\sqrt{\frac{n_1 n_2}{N}} \frac{\widehat{\Delta}^*(r, r)}{\sqrt{2}\lambda_r} \right)^2 \left(\frac{\lambda_r}{\widehat{\lambda}_m} \right)^2 + \sum_{1 \leq r < m \leq p} \left(\sqrt{\frac{n_1 n_2}{N}} \frac{\widehat{\Delta}^*(r, m)}{\lambda_r \lambda_m} \right)^2 \left(\frac{\lambda_r \lambda_m}{\widehat{\lambda}_r \widehat{\lambda}_m} \right)^2 \\ &\Rightarrow \chi_{p(p+1)/2}^2, \end{aligned}$$

since by assertion (12) we have that $\sqrt{n_1 n_2 / N} \widehat{\Delta}^*(r, r) / (\sqrt{2}\lambda_r)$ resp. $\sqrt{n_1 n_2 / N} \widehat{\Delta}^*(r, m) / (\lambda_r \lambda_m)$ are asymptotically independent, standard Gaussian distributed random variables, and, by Lemma 3 of Fremdt *et al.* (2012) and Lemma 6.1(i), we get that $\widehat{\lambda}_i^* \rightarrow \lambda_i$, in probability, for $i = 1, 2, \dots, p$. This completes the proof of the theorem. \square

Proof of Theorem 3.4: Define

$$Z_{n_1, n_2}^+(t) = \left[n_1^{-1/2} \sum_{j=1}^{n_1} \{X_{1,j}^+(t) - \overline{X}_N(t)\}, n_2^{-1/2} \sum_{j=1}^{n_2} \{X_{2,j}^+(t) - \overline{X}_N(t)\} \right], \quad t \in \mathcal{I},$$

and

$$Z_{i, n_i}^+(t) = n_i^{-1/2} \sum_{j=1}^{n_i} \{X_{i,j}^+(t) - \overline{X}_N(t)\}, \quad t \in \mathcal{I}, \quad i = 1, 2.$$

Notice that, conditionally on X_N , $Z_{1, n_1}^+(t)$ and $Z_{2, n_2}^+(t)$ are independent, have covariance operators $\widehat{\mathcal{C}}_1$ and $\widehat{\mathcal{C}}_2$, respectively, and $X_{1,j}^+(t)$ and $X_{2,j}^+(t)$ have the same mean function $\overline{X}_N(t)$. By a central limit theorem for triangular arrays of independent and identically distributed \mathcal{H} -valued random variables (see, e.g., Politis & Romano (1992, Theorem 4.2)), it follows that, conditionally on X_N , Z_{i, n_i}^+ converges weakly to a Gaussian random element \mathcal{U}_i with mean zero and covariance operator \mathcal{C}_i as $n_i \rightarrow \infty$.

By the independence of Z_{1, n_1}^+ and Z_{2, n_2}^+ , we have, conditionally on X_N ,

$$\begin{aligned} S_N^+ &= \frac{n_1 n_2}{N} \int_0^1 \{\overline{X}_{1, n_1}^+(t) - \overline{X}_{2, n_2}^+(t)\}^2 dt \\ &= \frac{n_1 n_2}{N} \int_0^1 \left[\frac{1}{n_1} \sum_{t=1}^{n_1} \{X_{1,j}^+(t) - \overline{X}_N(t)\} - \frac{1}{n_2} \sum_{t=1}^{n_2} \{X_{2,j}^+(t) - \overline{X}_N(t)\} \right]^2 dt \\ &= \frac{n_2}{N} \int_0^1 \{Z_{1, n_1}^+(t)\}^2 dt + \frac{n_1}{N} \int_0^1 \{Z_{2, n_2}^+(t)\}^2 dt + O_P(N^{-1/2}). \end{aligned}$$

From the above results, and taking into account that $n_1/N \rightarrow \theta_1$, we have that S_N^+ converges weakly to $(1 - \theta_1) \int_0^1 \Gamma_1^2(t) dt + \theta_1 \int_0^1 \Gamma_2^2(t) dt = \int_0^1 \Gamma^2(t) dt$ as $n_1, n_2 \rightarrow \infty$, and the assertion follows. \square

To prove Theorem 3.5, we first fix some notation. Let

$$\widehat{a}_i^+ = \langle \overline{X}_{1, n_1}^+ - \overline{X}_{2, n_2}^+, \widehat{\phi}_i^+ \rangle = \int_0^1 (\overline{X}_{1, n_1}^+(t) - \overline{X}_{2, n_2}^+(t)) \widehat{\phi}_i^+(t) dt, \quad i = 1, 2, \dots, p,$$

where $\hat{\phi}_i$, $i = 1, 2, \dots, p$ are the eigefunctions of $\hat{C}_N^+ = n_2/N\hat{C}_{1,n_1}^+ + n_1/N\hat{C}_{2,n_2}^+$ with $\hat{C}_{1,n_1}^+(t, s) = n_1^{-1} \sum_{j=1}^{n_1} (X_{1,j}^+(t) - \bar{X}_{1,n_1}(t))(X_{1,j}^+(s) - \bar{X}_{1,n_1}(s))$ and $\hat{C}_{2,n_2}^+(t, s) = n_2^{-1} \sum_{j=1}^{n_2} (X_{2,j}^+(t) - \bar{X}_{2,n_2}(t))(X_{2,j}^+(s) - \bar{X}_{2,n_2}(s))$. Let $\hat{\tau}_i^+$ be the eigenvalues corresponding to the eigenfunctions $\hat{\phi}_i^+$, $i = 1, 2, \dots, p$, of \hat{C}_N^+ .

The following lemma is proved along the same lines as Lemma 6.1 and is useful in establishing Theorem 3.5. Hence, its proof is omitted.

Lemma 6.3 *Under the assumptions of Theorem 3.5 we have, conditionally on \mathbf{X}_N that, for $1 \leq i \leq p$,*

$$(i) \quad |\hat{\tau}_i^+ - \hat{\tau}_i| = O_P(N^{-1/2}) \quad \text{and} \quad (ii) \quad \|\hat{\phi}_i^+ - \hat{c}_i^+ \hat{\phi}_i\| = O_P(N^{-1/2}),$$

where $\hat{c}_i^+ = \text{sign}(\langle \hat{\phi}_i^+, \hat{\phi}_i \rangle)$.

Proof of Theorem 3.5: Let $\hat{a}^+(1, p) = (\hat{a}_1^+, \hat{a}_2^+, \dots, \hat{a}_p^+)'$. We first show that

$$\mathcal{L}\left(\sqrt{\frac{n_1 n_2}{N}} \hat{a}^+(1, p) \middle| \mathbf{X}_N\right) \Rightarrow N(0, T), \quad (17)$$

where $T = (T(i, j))_{i,j=1,2,\dots,p}$ is a $p \times p$ diagonal matrix with $T(i, i) = \tau_i$. Let $\tilde{a}_i^+ = \langle \bar{X}_{1,n_1}^+ - \bar{X}_{2,n_2}^+, \hat{c}_i^+ \hat{\phi}_i \rangle$ and $\tilde{a}^+(1, p) = (\tilde{a}_1^+, \tilde{a}_2^+, \dots, \tilde{a}_p^+)'$. We have that

$$\begin{aligned} \sqrt{\frac{n_1 n_2}{N}} |\hat{a}_i^+ - \tilde{a}_i^+| &= \sqrt{\frac{n_1 n_2}{N}} |\langle \bar{X}_{1,n_1}^+ - \bar{X}_{2,n_2}^+, \hat{\phi}_i^+ - \hat{c}_i^+ \hat{\phi}_i \rangle| \\ &\leq \|\hat{\phi}_i^+ - \hat{c}_i^+ \hat{\phi}_i\| \left\| \sqrt{\frac{n_1 n_2}{N}} (\bar{X}_{1,n_1}^+ - \bar{X}_{2,n_2}^+) \right\| \\ &= O_P(\|\hat{\phi}_i^+ - \hat{c}_i^+ \hat{\phi}_i\|) \rightarrow 0, \end{aligned}$$

by Lemma 6.3(ii). Thus $\sqrt{n_1 n_2 / N} \hat{a}^+(1, p) = \sqrt{n_1 n_2 / N} \tilde{a}^+(1, p) + o_P(1)$. Now, let for $r \in \{1, 2\}$, $L_{r,j}^+(k) = \langle X_{r,j}^+ - \bar{X}_N, \hat{c}_k^+ \hat{\phi}_k \rangle$ and notice that

$$\sqrt{n_1 n_2 / N} \tilde{a}^+(1, p) = \left(\left(\sqrt{\frac{n_2}{N}} \frac{1}{\sqrt{n_1}} \sum_{j=1}^{n_1} L_{1,j}^+(k) - \sqrt{\frac{n_1}{N}} \frac{1}{\sqrt{n_2}} \sum_{j=1}^{n_2} L_{2,j}^+(k) \right), k = 1, 2, \dots, p \right).$$

Furthermore, the $L_{r,j}^+(k)$'s are independent and satisfy $E(L_{r,j}^+(k)) = 0$ and

$$\begin{aligned} \text{Cov}\left(n_r^{-1/2} \sum_{j=1}^{n_r} L_{r,j}^+(k_1), n_r^{-1/2} \sum_{j=1}^{n_r} L_{r,j}^+(k_2)\right) &= \mathbb{E}\left(L_{r,1}^+(k_1) L_{r,1}^+(k_2)\right) \\ &= \hat{c}_{k_1}^+ \hat{c}_{k_2}^+ \int_0^1 \int_0^1 \mathbb{E}(\varepsilon_{r,j}^+(t) \varepsilon_{r,j}^+(s)) \hat{\phi}_{k_1}(t) \hat{\phi}_{k_2}(s) dt ds \\ &= \hat{c}_{k_1}^+ \hat{c}_{k_2}^+ \int_0^1 \int_0^1 \hat{C}_{n,r}(t, s) \hat{\phi}_{k_1}(t) \hat{\phi}_{k_2}(s) dt ds. \end{aligned}$$

This implies that $E(\sqrt{n_1 n_2 / N} \tilde{a}^+(1, p)) = 0$ and that

$$\begin{aligned} \text{Cov}\left(\sqrt{n_1 n_2 / N} \tilde{a}_{k_1}^+, \sqrt{n_1 n_2 / N} \tilde{a}_{k_2}^+\right) &= \hat{c}_{k_1}^+ \hat{c}_{k_2}^+ \int_0^1 \int_0^1 \left(n_2 / N \hat{C}_{1, n_1}(t, s) + n_1 / N \hat{C}_{2, n_2}(t, s)\right) \hat{\phi}_{k_1}(t) \hat{\phi}_{k_2}(s) dt ds \\ &= \hat{c}_{k_1}^+ \hat{c}_{k_2}^+ \int_0^1 \int_0^1 \hat{C}_N(t, s) \hat{\phi}_{k_1}(t) \hat{\phi}_{k_2}(s) dt ds \\ &= \mathbf{1}_{\{k_1 = k_2\}} \hat{\tau}_{k_1} \rightarrow \tau_{k_1}. \end{aligned}$$

Hence, (17) follows then by a multivariate central limit theorem for triangular arrays of independent random variables.

Now, (17) and Lemma 6.3(i) lead to assertion (i) of the theorem, since

$$\begin{aligned} S_{p, N}^{+(1)} &= \frac{n_1 n_2}{N} \sum_{k=1}^p (\hat{a}_k^+)^2 / \hat{\tau}_k - \sum_{k=1}^p \frac{\hat{\tau}_k^+ - \hat{\tau}_k}{\hat{\tau}_k^+} \left(\sqrt{n_1 n_2 / N} \hat{a}_k^+ / \hat{\tau}_k \right)^2 \\ &= \frac{n_1 n_2}{N} \sum_{k=1}^p (\hat{a}_k^+)^2 / \hat{\tau}_k + O_P\left(\max_{1 \leq k \leq p} |\hat{\tau}_k^+ - \hat{\tau}_k|\right) \Rightarrow \chi_p^2, \end{aligned}$$

and to assertion (ii), since

$$S_{p, N}^{+(2)} = \sum_{k=1}^p \hat{\tau}_k \left(\sqrt{n_1 n_2 / N} \frac{\hat{a}_k^+}{\sqrt{\hat{\tau}_k}} \right)^2 \Rightarrow \sum_{k=1}^p \tau_k N_k^2,$$

where N_k , $k = 1, 2, \dots, p$, are independent, standard Gaussian random variables. This completes the proof of the theorem. \square

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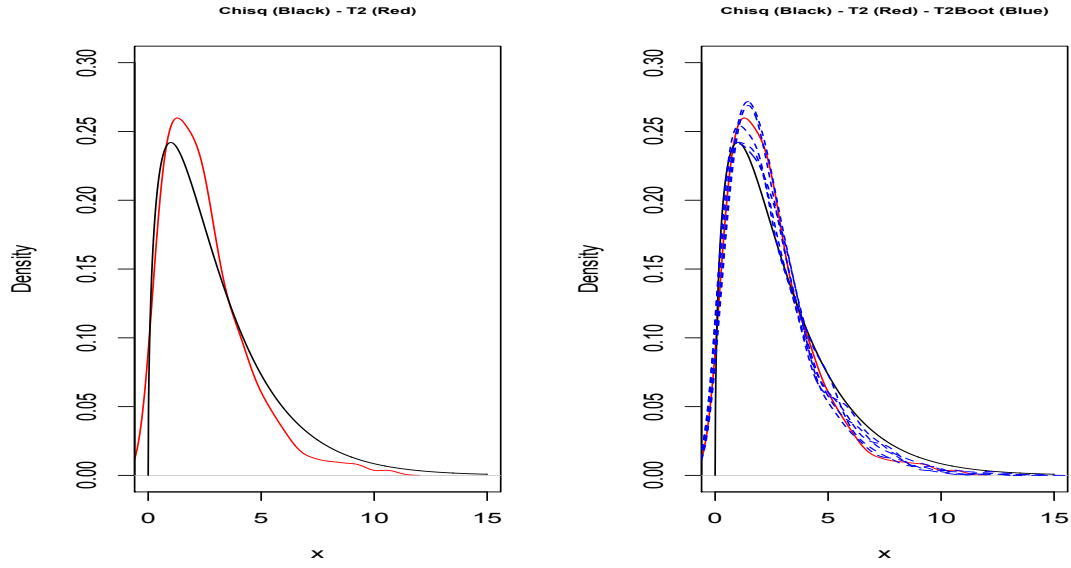


Figure 6.1: Density estimates, test statistic $T_{p,N}$ -Asym: $n_1 = n_2 = 25$, $p = 2$. Estimated exact density (red line), χ^2_3 approximation (black line) and 5 bootstrap approximations (dashed blue lines).

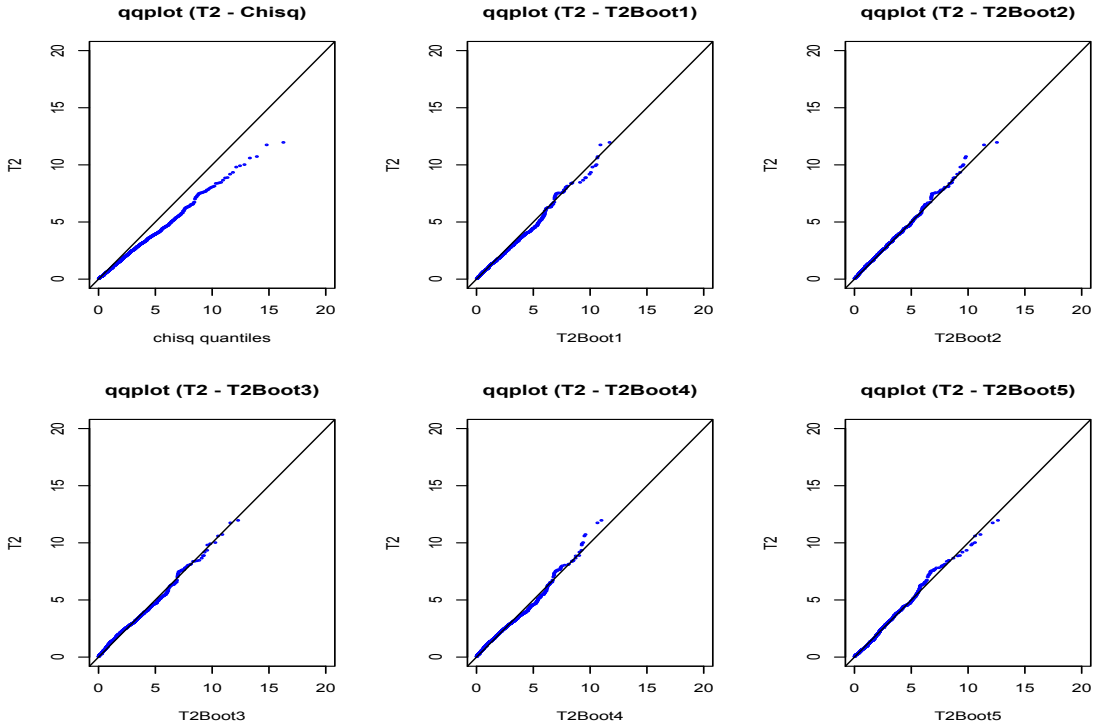


Figure 6.2: QQ-plots, Test statistic $T_{p,N}$ -Asym: $n_1 = n_2 = 25$, $p = 2$. Simulated exact distribution against the χ^2_3 distribution (qqplot (T2-Chisq)) and against 5 bootstrap approximations (qqplot(T2-T2Bootj), $j = 1, 2, \dots, 5$).

n_1	n_2	p	Test-Stat.	NG			BM		
				$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
25	25	2	$T_{p,N}$ -Asym	0.002	0.010	0.040	0.000	0.024	0.058
			$T_{p,N}$ -Boot	0.008	0.048	0.104	0.004	0.054	0.116
			$T_{p,N}^{(G)}$ -Asym	0.072	0.188	0.292	0.008	0.048	0.100
			$T_{p,N}^{(G)}$ -Boot	0.006	0.040	0.102	0.010	0.054	0.117
25	25	3	$T_{p,N}$ -Asym	0.000	0.010	0.034	0.000	0.020	0.048
			$T_{p,N}$ -Boot	0.006	0.046	0.096	0.003	0.032	0.068
			$T_{p,N}^{(G)}$ -Asym	0.068	0.198	0.320	0.006	0.048	0.106
			$T_{p,N}^{(G)}$ -Boot	0.008	0.046	0.094	0.004	0.038	0.088
25	25	–	T_N -Boot	0.006	0.034	0.112	0.004	0.060	0.134
50	50	2	$T_{p,N}$ -Asym	0.001	0.020	0.054	0.002	0.026	0.064
			$T_{p,N}$ -Boot	0.010	0.042	0.082	0.006	0.056	0.108
			$T_{p,N}^{(G)}$ -Asym	0.086	0.244	0.328	0.008	0.050	0.112
			$T_{p,N}^{(G)}$ -Boot	0.004	0.069	0.118	0.006	0.052	0.108
50	50	3	$T_{p,N}$ -Asym	0.002	0.012	0.050	0.000	0.016	0.046
			$T_{p,N}$ -Boot	0.006	0.052	0.092	0.012	0.042	0.094
			$T_{p,N}^{(G)}$ -Asym	0.124	0.254	0.340	0.006	0.048	0.094
			$T_{p,N}^{(G)}$ -Boot	0.004	0.040	0.114	0.008	0.042	0.093
50	50	–	T_N -Boot	0.010	0.056	0.088	0.016	0.052	0.110
100	100	2	$T_{p,N}$ -Asym	0.000	0.018	0.040	0.002	0.022	0.048
			$T_{p,N}$ -Boot	0.004	0.046	0.100	0.002	0.030	0.082
			$T_{p,N}^{(G)}$ -Asym	0.128	0.272	0.376	0.006	0.042	0.108
			$T_{p,N}^{(G)}$ -Boot	0.006	0.042	0.090	0.002	0.022	0.074
100	100	3	$T_{p,N}$ -Asym	0.004	0.018	0.044	0.002	0.034	0.060
			$T_{p,N}$ -Boot	0.006	0.054	0.094	0.002	0.037	0.075
			$T_{p,N}^{(G)}$ -Asym	0.146	0.312	0.410	0.008	0.048	0.090
			$T_{p,N}^{(G)}$ -Boot	0.006	0.042	0.100	0.009	0.050	0.092
100	100	–	T_N -Boot	0.006	0.030	0.096	0.006	0.052	0.108

Table 6.1: Empirical size of the tests for the equality of two covariance functions, based on the statistics $T_{p,N}^{(G)}$ -Asym and $T_{p,N}$ -Asym (asymptotic approximations) and $T_{p,N}^{(G)}$ -Boot, $T_{p,N}$ -Boot and T_N -Boot (bootstrap approximations), using two ($p = 2$) and three ($p = 3$) FPC's, both for Gaussian (BM) and non-Gaussian (NG) data. The curves in each sample were generated according to Brownian motions for Gaussian data and according to (8) for non-Gaussian data.

γ	Test-Stat.	N=M=25			N=M=50		
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
2.0	$T_{2,N}$ -Asym	0.000	0.042	0.220	0.074	0.610	0.850
	$T_{2,N}$ -Boot	0.008	0.188	0.448	0.398	0.834	0.940
	T_N -Boot	0.256	0.622	0.784	0.574	0.860	0.930
2.2	$T_{2,N}$ -Asym	0.000	0.058	0.318	0.196	0.774	0.932
	$T_{2,N}$ -Boot	0.022	0.328	0.594	0.696	0.938	0.980
	T_N -Boot	0.332	0.718	0.864	0.702	0.904	0.950
2.4	$T_{2,N}$ -Asym	0.000	0.132	0.440	0.358	0.902	0.976
	$T_{2,N}$ -Boot	0.044	0.460	0.732	0.818	0.976	0.990
	T_N -Boot	0.382	0.800	0.918	0.714	0.932	0.980
2.6	$T_{2,N}$ -Asym	0.000	0.162	0.532	0.500	0.946	0.988
	$T_{2,N}$ -Boot	0.082	0.596	0.840	0.900	0.990	0.996
	T_N -Boot	0.460	0.810	0.914	0.806	0.956	0.982
2.8	$T_{2,N}$ -Asym	0.000	0.228	0.658	0.652	0.980	0.996
	T_N -Boot	0.124	0.668	0.904	0.944	0.990	0.998
	T_N -Boot	0.462	0.838	0.934	0.822	0.946	0.986
3.0	$T_{2,N}$ -Asym	0.000	0.314	0.750	0.744	0.988	0.998
	$T_{2,N}$ -Boot	0.186	0.754	0.908	0.966	1.000	1.000
	T_N -Boot	0.530	0.844	0.942	0.834	0.956	0.976

Table 6.2: Empirical power of the tests for the equality of two covariance functions, based on the statistic $T_{p,N}$ -Asym (asymptotic approximation), $T_{p,N}$ -Boot and T_N -Boot (bootstrap approximation), using two ($p = 2$) FPC's, for non-Gaussian data. The curves in the first sample were generated according to (8) while the curves in the second sample were generated according to a scaled version of (8), i.e., $X_2(t) = \gamma X_1(t)$, $t \in \mathcal{I}$.

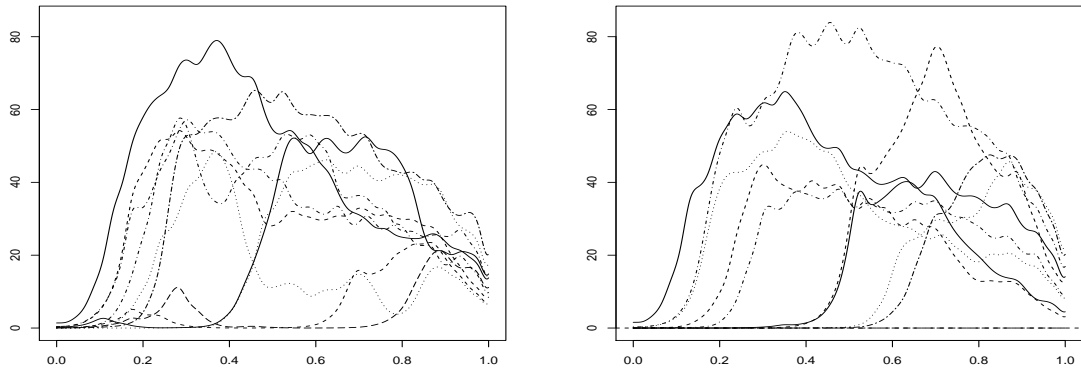


Figure 6.3: 10 randomly selected (smoothed) curves of short-lived (left panel) and 10 randomly selected (smoothed) curves of long-lived flies (right panel), scaled on the interval $\mathcal{I} = [0, 1]$.

n_1	n_2	p	Test-Stat.	NG			BB		
				$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
25	25	2	$S_{p,N}^{(1)}$ -Asym	0.016	0.062	0.148	0.016	0.062	0.132
			$S_{p,N}^{(1)}$ -Boot	0.012	0.052	0.122	0.006	0.050	0.110
			$S_{p,N}^{(2)}$ -Asym	0.012	0.086	0.142	0.018	0.060	0.124
			$S_{p,N}^{(2)}$ -Boot	0.016	0.042	0.084	0.018	0.060	0.132
			S_N -Boot	0.012	0.052	0.122	0.016	0.064	0.136
		3	$S_{p,N}^{(1)}$ -Asym	0.020	0.078	0.120	0.014	0.052	0.114
			$S_{p,N}^{(1)}$ -Boot	0.016	0.042	0.082	0.014	0.052	0.100
			$S_{p,N}^{(2)}$ -Asym	0.012	0.048	0.108	0.006	0.046	0.100
			$S_{p,N}^{(2)}$ -Boot	0.008	0.036	0.072	0.018	0.052	0.120
			S_N -Boot	0.016	0.042	0.082	0.018	0.056	0.118
50	50	2	$S_{p,N}^{(1)}$ -Asym	0.018	0.060	0.114	0.008	0.066	0.130
			$S_{p,N}^{(1)}$ -Boot	0.008	0.036	0.082	0.006	0.044	0.086
			$S_{p,N}^{(2)}$ -Asym	0.008	0.056	0.114	0.008	0.068	0.114
			$S_{p,N}^{(2)}$ -Boot	0.010	0.048	0.112	0.008	0.048	0.098
			S_N -Boot	0.010	0.048	0.112	0.008	0.046	0.100
		3	$S_{p,N}^{(1)}$ -Asym	0.026	0.088	0.134	0.016	0.070	0.116
			$S_{p,N}^{(1)}$ -Boot	0.010	0.036	0.090	0.012	0.034	0.070
			$S_{p,N}^{(2)}$ -Asym	0.010	0.044	0.100	0.024	0.068	0.124
			$S_{p,N}^{(2)}$ -Boot	0.016	0.054	0.110	0.060	0.052	0.102
			S_N -Boot	0.016	0.054	0.110	0.008	0.056	0.106
100	100	2	$S_{p,N}^{(1)}$ -Asym	0.014	0.064	0.120	0.006	0.038	0.098
			$S_{p,N}^{(1)}$ -Boot	0.006	0.042	0.086	0.010	0.050	0.108
			$S_{p,N}^{(2)}$ -Asym	0.014	0.068	0.122	0.002	0.042	0.098
			$S_{p,N}^{(2)}$ -Boot	0.006	0.058	0.096	0.006	0.044	0.122
			S_N -Boot	0.006	0.058	0.096	0.006	0.042	0.120
		3	$S_{p,N}^{(1)}$ -Asym	0.014	0.062	0.108	0.016	0.062	0.122
			$S_{p,N}^{(1)}$ -Boot	0.006	0.046	0.096	0.014	0.066	0.120
			$S_{p,N}^{(2)}$ -Asym	0.014	0.062	0.110	0.016	0.076	0.136
			$S_{p,N}^{(2)}$ -Boot	0.010	0.048	0.092	0.016	0.068	0.112
			S_N -Boot	0.010	0.048	0.092	0.016	0.070	0.112

Table 6.3: Empirical size of the tests for the equality of two mean functions, based on the statistics $S_{p,N}^{(1)}$ -Asym and $S_{p,N}^{(2)}$ -Asym (asymptotic approximations) and $S_{p,N}^{(1)}$ -Boot, $S_{p,N}^{(2)}$ -Boot and S_N -Boot (bootstrap approximations), using two ($p = 2$) and three ($p = 3$) FPC's, both for Gaussian and non-Gaussian data. The curves in each sample were generated according to Brownian Bridges for Gaussian data (BB) and according to (8) for non-Gaussian data (NG).

δ	Test-Stat.	N=M=25			N=M=50		
		$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$	$\alpha = 0.01$	$\alpha = 0.05$	$\alpha = 0.10$
1.0	$S_{2,N}^{(1)}$ -Asym	0.088	0.186	0.304	0.142	0.314	0.450
	$S_{2,N}^{(1)}$ -Boot	0.052	0.160	0.260	0.122	0.308	0.422
	$S_{2,N}^{(2)}$ -Asym	0.114	0.264	0.354	0.188	0.410	0.522
	$S_{2,N}^{(2)}$ -Boot	0.100	0.226	0.352	0.218	0.402	0.516
	S_N -Boot	0.112	0.270	0.392	0.264	0.470	0.610
1.2	$S_{2,N}^{(1)}$ -Asym	0.138	0.280	0.404	0.198	0.422	0.550
	$S_{2,N}^{(1)}$ -Boot	0.076	0.232	0.328	0.162	0.372	0.524
	$S_{2,N}^{(2)}$ -Asym	0.168	0.332	0.458	0.296	0.524	0.648
	$S_{2,N}^{(2)}$ -Boot	0.178	0.338	0.440	0.262	0.524	0.656
	S_N -Boot	0.202	0.382	0.500	0.336	0.630	0.762
1.4	$S_{2,N}^{(1)}$ -Asym	0.170	0.356	0.466	0.360	0.564	0.676
	$S_{2,N}^{(1)}$ -Boot	0.118	0.264	0.378	0.284	0.510	0.654
	$S_{2,N}^{(2)}$ -Asym	0.238	0.424	0.530	0.430	0.676	0.750
	$S_{2,N}^{(2)}$ -Boot	0.214	0.416	0.546	0.432	0.662	0.758
	S_N -Boot	0.246	0.496	0.616	0.536	0.776	0.878
1.6	$S_{2,N}^{(1)}$ -Asym	0.262	0.448	0.568	0.454	0.650	0.756
	$S_{2,N}^{(1)}$ -Boot	0.134	0.350	0.484	0.408	0.638	0.758
	$S_{2,N}^{(2)}$ -Asym	0.296	0.516	0.640	0.558	0.740	0.830
	$S_{2,N}^{(2)}$ -Boot	0.284	0.516	0.632	0.558	0.766	0.868
	S_N -Boot	0.358	0.614	0.728	0.686	0.888	0.940
1.8	$S_{2,N}^{(1)}$ -Asym	0.302	0.558	0.662	0.572	0.772	0.852
	$S_{2,N}^{(1)}$ -Boot	0.196	0.440	0.578	0.520	0.754	0.842
	$S_{2,N}^{(2)}$ -Asym	0.404	0.634	0.722	0.672	0.858	0.918
	$S_{2,N}^{(2)}$ -Boot	0.380	0.610	0.708	0.686	0.866	0.914
	S_N -Boot	0.458	0.698	0.820	0.814	0.942	0.982
2.0	$S_{2,N}^{(1)}$ -Asym	0.380	0.574	0.690	0.668	0.824	0.880
	$S_{2,N}^{(1)}$ -Boot	0.286	0.512	0.666	0.642	0.832	0.906
	$S_{2,N}^{(2)}$ -Asym	0.434	0.658	0.776	0.728	0.888	0.936
	$S_{2,N}^{(2)}$ -Boot	0.458	0.680	0.794	0.792	0.914	0.940
	S_N -Boot	0.576	0.798	0.880	0.900	0.974	0.994

Table 6.4: Empirical power of the tests for the equality of two mean functions, based on the statistics $S_{p,N}^{(1)}$ -Asym and $S_{p,N}^{(2)}$ -Asym (asymptotic approximations) and $S_{p,N}^{(1)}$ -Boot, $S_{p,N}^{(2)}$ -Boot and S_N -Boot (bootstrap approximations), using two ($p = 2$) FPC's, for non-Gaussian data. The curves in the two samples were generated according to the model $X_i(t) = \mu_i(t) + \epsilon_i(t)$ with $\epsilon_i(t)$ distributed independently according to (8), for $i \in \{1, 2\}$, $t \in \mathcal{I}$. The mean functions were set equal to $\mu_1(t) = 0$ and $\mu_2(t) = \delta$.

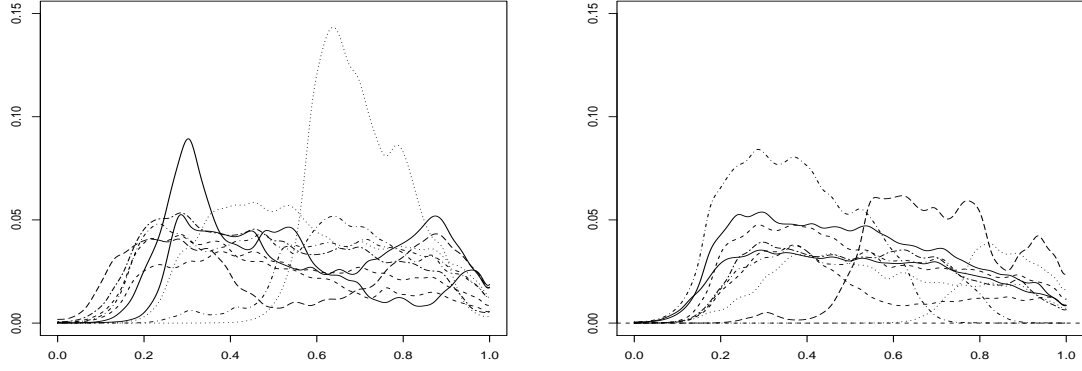


Figure 6.4: 10 randomly selected (smoothed) curves of short-lived (left panel) and 10 randomly selected (smoothed) curves of long-lived flies (right panel) - relative to the number of eggs laid in the fly's lifetime - scaled on the interval $\mathcal{I} = [0, 1]$.

T_N -Boot	Absolute					Relative				
	0.179					0.003				
	$p = 4$	$p = 5$	$p = 6$	$p = 7$	$p = 8$	$p = 5$	$p = 6$	$p = 7$	$p = 8$	$p = 9$
$T_{p,N}$ -Asym	0.253	0.211	0.385	0.545	0.520	0.004	0.021	0.064	0.130	0.121
$T_{p,N}$ -Boot	0.187	0.152	0.315	0.481	0.460	0.001	0.005	0.016	0.072	0.069
$T_{p,N}^{(G)}$ -Asym	0.090	0.038	0.058	0.020	0.009	0.000	0.000	0.000	0.000	0.000
$T_{p,N}^{(G)}$ -Boot	0.136	0.092	0.145	0.101	0.100	0.025	0.001	0.001	0.001	0.002
f_p	0.940	0.958	0.974	0.982	0.989	0.845	0.912	0.949	0.974	0.985

Table 6.5: p -values of the tests for the equality of covariance functions, based on the statistics $T_{p,N}^{(G)}$ -Asym and $T_{p,N}$ -Asym (asymptotic approximations), $T_{p,N}^{(G)}$ -Boot, $T_{p,N}$ -Boot and T_N -Boot (bootstrap approximations), applied to absolute (left panel) and relative (right panel) egg-laying curves. The term f_p denotes the fraction of the sample variance explained by the first p FPC's, i.e., $f_p = (\sum_{k=1}^p \hat{\lambda}_k) / (\sum_{k=1}^N \hat{\lambda}_k)$.

S_N -Boot	0.011							
	$p = 2$	$p = 3$	$p = 4$	$p = 5$	$p = 6$	$p = 7$	$p = 8$	$p = 9$
$S_{p,N}^{(1)}$ -Asym	0.021	0.029	0.056	0.099	0.154	0.054	0.025	0.040
$S_{p,N}^{(1)}$ -Boot	0.020	0.035	0.067	0.108	0.172	0.070	0.035	0.051
$S_{p,N}^{(2)}$ -Asym	0.007	0.008	0.009	0.009	0.010	0.010	0.010	0.010
$S_{p,N}^{(2)}$ -Boot	0.012	0.011	0.011	0.011	0.012	0.011	0.011	0.011
f_p	0.837	0.899	0.939	0.958	0.973	0.982	0.989	0.994

Table 6.6: p -values of the tests for the equality of mean functions, based on the statistics $S_{p,N}^{(1)}$ -Asym and $S_{p,N}^{(2)}$ -Asym (asymptotic approximations), $S_{p,N}^{(1)}$ -Boot, $S_{p,N}^{(2)}$ -Boot and S_N -Boot (bootstrap approximations), applied to absolute egg-laying curves. The term f_p denotes the fraction of the sample variance explained by the first p FPC's, i.e., $f_p = (\sum_{k=1}^p \hat{\lambda}_k) / (\sum_{k=1}^N \hat{\lambda}_k)$.